

A New Proof to the Period Problems of $GL(2)$

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Abstract We use the relations between the base change representations, theta lifts and Whittaker model, to give a new proof to the period problems of $GL(2)$ over a quadratic local field extension E/F . And we classify both local and global $D^\times(F)$ -distinguished representations π^D of $D^\times(E)$, where D^\times is an inner form of GL_2 defined over a nonarchimedean field or a number field F .

Keywords theta lifts, periods, inner forms, Whittaker model

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1 Introduction

Period problems, which are closely related to Harmonic Analysis, have been extensively studied for classical groups. The most general situations have been studied in [SV12], and we will focus on the period problems of $GL(2)$ in this paper.

Assume F is a nonarchimedean local field of characteristic 0. Let G be a connected reductive group defined over F and H be a subgroup of G . Given a smooth irreducible representation π of $G(F)$, one may consider $\dim \operatorname{Hom}_{H(F)}(\pi, \mathbb{C})$. If it is nonzero, then we say that π is $H(F)$ -distinguished, or has a nonzero $H(F)$ -period. One may also consider the Ext version $\operatorname{Ext}_{H(F)}^1(\pi, \mathbb{C})$ in the category of smooth representations of $H(F)$.

Let W_F be the Weil group of F and WD_F be the Weil-Deligne group. Assume τ is an irreducible smooth representation of $GL_2(F)$, with Langlands parameter $\phi_\tau : WD_F \rightarrow GL_2(\mathbb{C})$. Assume E is a quadratic field extension of F . Then $\phi_\tau|_{WD_E}$ gives an admissible representation of $GL_2(E)$, which is denoted by $BC(\tau)$. Then we have the following results:

Main Theorem (Local) Let E be a quadratic field extension of a nonarchimedean local field F , with the Galois group $\operatorname{Gal}(E/F) = \{1, \sigma\}$ and an associated quadratic character $\omega_{E/F}$ of F^\times . Assume π is a generic irreducible smooth representation of $GL_2(E)$, with the central character ω_π and $\omega_\pi|_{F^\times} = 1$.

(1) The following statements are equivalent:

- (i) π is $GL_2(F)$ -distinguished;
- (ii) $\pi = BC(\tau) \otimes \mu^{-1}$ for some irreducible representation τ of $GL_2(F)$, where $\omega_\pi = \mu^\sigma / \mu$ and $\omega_\tau = \omega_{E/F} \cdot \mu|_{F^\times}$;

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- (iii) the Langlands parameter $\phi_\pi : WD_E \rightarrow GL_2(\mathbb{C})$ is conjugate-orthogonal in the sense of [GGP12, Section 3].
- (2) Assume D is the nonsplit quaternion algebra defined over F , then $D^\times(E) = GL_2(E)$. Then the following statements are equivalent:
 - (i) π is $D^\times(F)$ -distinguished;
 - (ii) π is $GL_2(F)$ -distinguished and $\pi \neq \pi(\chi_1, \chi_2)$, where $\chi_1 \neq \chi_2$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$.
- (3) Regarding $\pi|_{D^\times(F)}$ (resp. $\pi|_{GL_2(F)}$) as a smooth representation of $PD^\times(F)$ (resp. $PGL_2(F)$), then the following statements are equivalent:
 - (i) $\dim Hom_{PGL_2(F)}(\pi|_{GL_2(F)}, \mathbb{C}) - \dim Ext_{PGL_2(F)}^1(\pi|_{GL_2(F)}, \mathbb{C}) = 1$;
 - (ii) $\dim Hom_{PD^\times(F)}(\pi|_{D^\times(F)}, \mathbb{C}) - \dim Ext_{PD^\times(F)}^1(\pi|_{D^\times(F)}, \mathbb{C}) = 1$;
 - (iii) π is $D^\times(F)$ -distinguished.

For both (1) and (2), Flicker [Fli91] proved the cases for the principal series, and later Flicker and Hakim [FH94] proved the cases for discrete series, using relative trace formula. In this paper, we use the local theta correspondence to give a new proof. Then we follow Prasad's spirit in [Pra13] to consider the Ext version and get (3).

If F is a number field, let $\mathbb{A} = \mathbb{A}_F$ be the adele ring of F . Let G be a connected reductive group defined over F and H be a subgroup of G . Assume π is an automorphic representation of $G(\mathbb{A})$. Let $Z(\mathbb{A})$ be the center of $H(\mathbb{A})$. If the integral

$$P(\phi) = \int_{H(F)Z(\mathbb{A}) \backslash H(\mathbb{A})} \phi(h) dh, \quad \text{where } \phi \in \pi$$

converges and is nonzero on π , then we say that π is H -distinguished. In [Fli88], Flicker showed that a cuspidal automorphic representation π of $GL_n(\mathbb{A}_E)$, where E is a quadratic extension of F , is $GL_n(\mathbb{A})$ -distinguished if and only if the partial Asai L-function $L^S(s, \pi, As)$ has a pole at $s = 1$. In this paper, we will use the global theta lift and its Whittaker model to discuss the case $G(\mathbb{A}) = D^\times(\mathbb{A}_E)$ and $H(\mathbb{A}) = D^\times(\mathbb{A})$, where D^\times is an inner form of GL_2 defined over F .

By the Galois cohomology, the inner form D^\times of GL_2 corresponds to a quaternion algebra defined over F with involution $*$, denoted by D . Then the tensor product $D \otimes_F E$ is an even Clifford algebra of 8 dimensions defined over F , denoted by B . There are two natural F -linear automorphisms on B , one is induced by the involution $*$, and the other is induced by the Galois action. More precisely, for the element $(d \otimes e) \in B$, we define

$$(d \otimes e)^* = d^* \otimes e, \quad (d \otimes e)^\sigma = d \otimes \sigma(e).$$

Then D is the set consisting of all Galois invariant elements and E coincides with the set consisting of all fixed points under the involution. Moreover, the intersection of D and E coincides with F . Set $X_D = \{x \in B \mid x^\sigma = x^*\}$, which is a 4-dimensional quadratic vector space defined over F , with a quadratic form $q(x) = xx^*$ taking values in E . In fact, the element $q(x)$ is Galois invariant, then $q(x)$ lies inside F and the quadratic form is well-defined on the space X_D . Now we can define an $(F^\times \times B^\times)$ -action on X_D to be

$$(\lambda, g).x = \lambda g x \sigma(g)^*,$$

where $\lambda \in F^\times$, $g \in B^\times$. In fact, the quaternion algebra $D(E)$ over E coincides with the even Clifford algebra B . In [Rob01], Roberts showed that the similitude group $GSO(X_D, q)$ is a quotient of $F^\times \times D^\times(E)$, and the special orthogonal group $SO(Y, F)$ for some 3-dimensional subspace Y in X_D is $PD^\times(F)$.

Given any cuspidal automorphic representation π^D of $D^\times(\mathbb{A}_E)$, we may consider the Jacquet-Langlands correspondence representation π of $GL_2(\mathbb{A}_E)$. Assume the central character $\omega_\pi = \mu^\sigma / \mu$ for some grossencharacter μ of \mathbb{A}_E^\times . Then $(\pi \otimes \mu) \boxtimes \mu|_{F^\times}$ is an automorphic representation of $GSO(V, \mathbb{A})$, where $V = E \oplus \mathbb{H}$ is a quadratic space over F .

Main Theorem (Global) Assume F is a number field and E/F is a quadratic field extension. Let D^\times be an inner form of GL_2 defined over F . Let π^D be a cuspidal automorphic representation of $D^\times(\mathbb{A}_E)$, with $\omega_{\pi^D}|_{Z(\mathbb{A})} = 1$. Assume the Jacquet-Langlands correspondence representation π is a cusp form of $GL_2(\mathbb{A}_E)$. Then the following statements are equivalent:

- (i) π^D is $D^\times(\mathbb{A})$ -distinguished;
- (ii) $\Sigma^D = (\pi^D \otimes \mu) \boxtimes \mu|_{\mathbb{A}^\times}$ as a cuspidal automorphic representation of $GSO(X_D, \mathbb{A})$ is $PD^\times(\mathbb{A})$ -distinguished for some Hecke character $\mu : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$;
- (iii) $\Sigma = (\pi \otimes \mu) \boxtimes \mu|_{\mathbb{A}^\times}$ as a cuspidal automorphic representation of $GSO(V, \mathbb{A})$ is $PGL_2(\mathbb{A})$ -distinguished and for the local place v of F , where D_v is ramified and E_v is a field, the local representation π_v is isomorphic to $BC(\tau_v) \otimes \mu_v^{-1}$ for some representation τ_v of $GL_2(F_v)$ with $\omega_{\tau_v} = \mu_v|_{F_v^\times} \omega_{E_v/F_v}$ and $\tau_v|_{GL_2^+}$ is irreducible, where $GL_2^+ = \{g \in GL_2(F_v) | \omega_{E_v/F_v}(\det(g)) = 1\}$.

In a short summary, we give an answer to the local problems in [Pra15, Conjecture 2] when $G = GL_2$ with trivial character, which is well-known in [FH94]. But we use a different method, i.e. the theta correspondence and base change to recover the local period problems for both GL_2 and D^\times . Then the Ext version for the Branch law can be obtained easily via the Mackey theory. In the global situation, we use the regularized Siegel-Weil formula in the first term range in [GT11a] and the global theta lifts from orthogonal groups to show [FH94, Theorem 2], i.e. our main global theorem. However, our method can not be extended to the cases for the general inner forms of GL_n in [FH94, Theorem 5].

Now we briefly describe the contents and the organization of this paper. In section 2 and 3, we set up the notations about the quadratic vector spaces and local theta lifts. In section 4, the Jacquet modules and the local theta lifts are computed explicitly. In section 5, we give the proof for the **Main Theorem (Local)**. In section 6, the global theta lifts are introduced. In section 7, we give the proof of the **Main Theorem (Global)**.

2 Preliminaries

We follow the notations in [Rob01] to introduce the 4-dimensional quadratic spaces. Let F be a field with characteristic not equal to 2, and let E be a quadratic extension over F with Galois group $Gal(E/F) = \{1, \sigma\}$. Assume D is a non-split quaternion algebra defined over F with an involution $*$. Then $B = D \otimes_F E$ is an algebra over F , with Galois action σ and involution $*$. Set $X_D = \{x \in B | x^* = x^\sigma\}$ be a 4-dimensional vector space with a quadratic form q taking values in F .

Now we define an $(F^\times \times B^\times)$ -action on X_D , for $\lambda \in F^\times$ and $g \in B^\times$, define

$$(\lambda, g)x = \lambda g x \sigma(g)^*.$$

This action preserves X_D , and $q((\lambda, g)x) = \lambda^2 N_{E/F}(gg^*) \cdot q(x)$. Thus, the element (λ, g) lies inside the similitude orthogonal group $GO(X_D, F)$, with similitude factor $\lambda^2 N(gg^*)$. In fact, it lies in the similitude special orthogonal group $GSO(X_D, F)$.

Theorem 2.1. [Rob01] *There is an exact sequence*

$$1 \longrightarrow E^\times \longrightarrow F^\times \times B^\times \longrightarrow GSO(X_D, F) \longrightarrow 1,$$

where the first arrow is $e \mapsto (N_{E/F}(e^{-1}), e)$.

Let us consider the vector $1 \otimes 1 \in X_D$, denoted by y . For arbitrary $d \in D^\times$, the element $(N_{D/F}(d^{-1}), d) \in F^\times \times B^\times$ fixes the vector y . Set y^\perp to be the orthogonal subspace with respect to y in the quadratic space X_D .

Theorem 2.2. [Rob01] *Assume D is a quaternion algebra over F , let y, X_D, B be the notations as above. Then there exist an exact sequence*

$$1 \rightarrow F^\times \rightarrow D^\times \rightarrow SO(y^\perp, F) \rightarrow 1,$$

and the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & F^\times & \longrightarrow & D^\times & \longrightarrow & SO(y^\perp, F) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \longrightarrow & GSO(X_D, F) \longrightarrow 1, \end{array}$$

where the inclusion from D^\times to $F^\times \times B^\times$ is given by $g \mapsto (N_{D/F}(g^{-1}), g)$.

Assume X (resp. Y) is a 4-dimensional quadratic space over F with a quadratic form q_X (resp. q_Y), pick one anisotropic vector $v_0 \in X$ with $q_X(v_0) \neq 0$. Assume L is an anisotropic line in Y , we define that a pair (Y, L) is isomorphic to the pair (X, Fv_0) if there is an invertible linear transform $T : X \rightarrow Y$ such that $T^*(q_Y) = \lambda q_X$ for some $\lambda \in F^\times$ and $T(v_0) \in L$.

Lemma 2.3. *The followings are equivalent:*

- (i) *the pair (D, E) , where D is a quaternion algebra defined over F , and E is a separable quadratic extension algebra over F ;*
- (ii) *the 3-dimensional quadratic space class W over F ;*
- (iii) *the 4-dimensional quadratic space class (X, q_X) with a fixed anisotropic vector v_0 such that $q(v_0) = 1$;*
- (iv) *the triple class (Y, q_Y, L) where L is an anisotropic line in Y .*

Proof. (i) \Leftrightarrow (ii) comes from the Galois cohomology. Since $O(3) = SO(3) \times \mu_2$, then

$$H^1(F, O(3)) \cong H^1(F, \mu_2) \times H^1(F, PGL_2) \cong F^\times / F^{\times 2} \times Br_2(F).$$

The Brauer group $Br_2(F)$ corresponds to quaternion division algebras over F , and the quotient group $F^\times / F^{\times 2}$ corresponds to the separable quadratic algebras over F .

(ii) \Leftrightarrow (iii) is easy to see if we choose $W = \{v \in X : q(v + v_0) = q(v) + q(v_0)\}$.

(iii) is equivalent to (iv) by definition. \square

If F is a number field, and E is a quadratic extension of F . Let D be a quaternion algebra defined over F . Let $S_{D,E}$ be the set of places v of F such that D_v is ramified and v splits in E .

Proposition 2.4. *Assume D and D' are quaternion algebras over F . Then $D \otimes E \cong D' \otimes E$ as E -algebras if and only if $S_{D,E} = S_{D',E}$.*

To close this section, we give one example.

Example 2.5. *Assume F is a local field of characteristic zero. Let E be a quadratic extension of F with Galois group $Gal(E/F) = \{1, \sigma\}$. Assume $D_1(F) \cong M_2(F)$ is the split quaternion algebra. Let D_2 be the unique nonsplit quaternion algebra defined over F . Then $D_1(E) \cong D_2(E) \cong M_2(E)$ with different Galois actions σ_1, σ_2 . Set $\epsilon \in F^\times \setminus NE^\times$, define*

$$\sigma_1 \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \sigma(e) & \sigma(f) \\ \sigma(g) & \sigma(h) \end{pmatrix}, \text{ and } \sigma_2 \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma(e) & \sigma(f) \\ \sigma(g) & \sigma(h) \end{pmatrix} \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}^{-1}.$$

Moreover, $D_2(F)$ can be regarded as $\left\{ \begin{pmatrix} e & f\epsilon \\ \sigma(f) & \sigma(e) \end{pmatrix} \middle| e, f \in E \right\} \subset M_2(E)$. And the involution is $\begin{pmatrix} e & f \\ g & h \end{pmatrix}^* = \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$, the quadratic spaces are

$$X_{D_1} = \left\{ \begin{pmatrix} f & e \\ g & \sigma(f) \end{pmatrix} \middle| e, f, g \in E, \sigma(e) = -e, \sigma(g) = -g \right\},$$

and $X_{D_2} = \left\{ \begin{pmatrix} f & -\epsilon e \\ \sigma(e) & g \end{pmatrix} \middle| f, g \in F, e \in E \right\}$, with an anisotropic line Fy , where $y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3 The Local Theta Correspondences For Similitudes

In this section, we will briefly recall some results about the local theta correspondence for similitude groups. There are many references, such as [Kud96], [Rob01] and [GT11b].

Let F be a local field of characteristic zero. Consider the dual pair $O(V) \times Sp(W)$. For simplicity, we may assume that $\dim V$ is even. Fix a nontrivial additive character ψ of F . Let ω_ψ be the Weil representation for $O(V) \times Sp(W)$, which can be describe as follows. Fix a Witt decomposition $W = X \oplus Y$ and let $P(Y) = GL(Y)N(Y)$ be the parabolic subgroup stabilizing the maximal isotropic subspace Y . Then

$$N(Y) = \{b \in Hom(X, Y) \mid b^t = b\},$$

where $b^t \in \text{Hom}(Y^*, X^*) \cong \text{Hom}(X, Y)$. The Weil representation ω_ψ can be realized on the Schwartz space $S(X \otimes V)$ and the action of $P(Y) \times O(V)$ is given by the usual formula

$$\begin{cases} \omega_\psi(h)\phi(x) = \phi(h^{-1}x), & \text{for } h \in O(V), \\ \omega_\psi(a)\phi(x) = \chi_V(\det_Y(a))|\det_Y a|^{\frac{1}{2}\dim V}\phi(a^{-1} \cdot x), & \text{for } a \in GL(Y), \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x \rangle)\phi(x), & \text{for } b \in N(Y), \end{cases}$$

where χ_V is the quadratic character associated to $\text{disc} V \in F^\times/F^{\times 2}$ and $\langle -, - \rangle$ is the natural symplectic form on $W \otimes V$. To describe the full action of $Sp(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $O(V)$ (resp. $Sp(W)$), the maximal π -isotypic quotient has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation of $Sp(W)$ (resp. $O(V)$). We call $\Theta_\psi(\pi)$ the big theta lift of π . It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$, which is called the small theta lift of π . Then there is a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is non-zero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger when the residual characteristic p of F is not 2. Recently, it has been proved completely, see [GT16a],[GT16b] and [GS15].

Theorem 3.1. *The Howe conjecture holds.*

We now extend Weil representation to the case of similitude groups. Let λ_V and λ_W be the similitude factors of $GO(V)$ and $GSp(W)$ respectively. We shall consider the group

$$R = GO(V) \times GSp^+(W)$$

where $GSp^+(W)$ is the subgroup of $GSp(W)$ consisting of elements g such that $\lambda_W(g)$ lies in the image of λ_V . Define

$$R_0 = \{(h, g) \in R \mid \lambda_V(h)\lambda_W(g) = 1\}$$

to be the subgroup of R . The Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(g, h)\phi = |\lambda_V(h)|^{-\frac{1}{8}\dim V \cdot \dim W} \omega(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda_W(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in Sp(W).$$

Here the central elements $(t, t^{-1}) \in R_0$ acts by the quadratic character $\chi_V(t)^{\frac{\dim W}{2}}$, which is slightly different from the normalization used in [Rob01].

Now we consider the compactly induced representation

$$\Omega = \text{ind}_{R_0}^R \omega_\psi.$$

As a representation of R , Ω depends only on the orbit of ψ under the evident action of $\text{Im}\lambda_V \subset F^\times$. For example, if λ_V is surjective, then Ω is independent of ψ . For any irreducible representation π or $GO(V)$ (resp. $GSp^+(W)$), the maximal π -isotropic quotient of Ω has the form

$$\pi \otimes \Theta_\psi(\pi)$$

where $\Theta_\psi(\pi)$ is some smooth representation of $GSp^+(W)$ (resp. $GO(V)$). Similarly, we let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$. Note that though $\Theta_\psi(\pi)$ may be reducible, it has a central character $\omega_{\Theta_\psi(\pi)}$ given by

$$\omega_{\Theta_\psi(\pi)} = \chi_V^{\frac{\dim W}{2}} \omega_\pi.$$

There is an extended Howe conjecture for similitude groups, which says that $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is non-zero and the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain. It was shown by Roberts [Rob96] that this follows from the Howe conjecture for isometry groups.

Sometimes, we use $\theta_\psi^*(\pi)$ to denote the theta lift of π from $GSp(W)$ to $GO(V)$, or even $\theta_\psi(\pi)$. Then $\theta_\psi^*(\pi) = \theta_\psi^*(\pi \otimes \mu)$ where $\mu : GSp(W)/GSp^+(W) \rightarrow \langle \pm 1 \rangle$ is a quadratic character.

First Occurrence Indices for pairs of orthogonal Witt Towers Let W_n be the $2n$ -dimensional symplectic vector space with associated symplectic group $Sp(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with nontrivial discriminant. More precisely, let

$$V_r^+ = V_E \oplus \mathbb{H}^{r-1} \quad \text{and} \quad V_r^- = \epsilon V_E \oplus \mathbb{H}^{r-1}$$

and denote the orthogonal groups by $O(V_r^+)$ and $O(V_r^-)$ respectively. For an irreducible representation π of $Sp(W_n)$, one may consider the theta lifts $\theta_r^+(\pi)$ and $\theta_r^-(\pi)$ to $O(V_r^+)$ and $O(V_r^-)$ respectively, with respect to a fixed non-trivial additive character ψ . Set

$$\begin{cases} r^+(\pi) = \inf\{2r : \theta_r^+(\pi) \neq 0\}; \\ r^-(\pi) = \inf\{2r : \theta_r^-(\pi) \neq 0\}. \end{cases}$$

Then Kudla, Rallis, B. Sun and C. Zhu showed:

Theorem 3.2. [KR05][SZ15] [Conservation Relation] For any irreducible representation π of $Sp(W_n)$, we have

$$r^+(\pi) + r^-(\pi) = 4n + 4 = 4 + 2 \dim W_n.$$

On the other hand, one may consider the mirror situation, where one fixes an irreducible representation of $O(V_r^+)$ or $O(V_r^-)$ and consider its theta lifts $\theta_n(\pi)$ to the tower of symplectic group $Sp(W_n)$. Then with $n(\pi)$ defined in the analogous fashion, we have

$$n(\pi) + n(\pi \otimes \det) = \dim V_r^\pm.$$

4 Computation for Local Theta Lifts

In this section, our aim is to compute the local theta lifts explicitly. Let F be a p -adic local field, \mathcal{O}_F be the integer ring with a unique prime ideal \mathfrak{p} . Let E/F be a quadratic field extension with Galois group $\text{Gal}(E/F) = \{1, \sigma\}$. Let $\omega_{E/F} : F^\times \rightarrow \mathbb{C}^\times$ be the quadratic character associated with E by the local class field theory. Let W be a 2-dimensional symplectic vector space over F . There are two types of 4-dimension quadratic spaces with discriminant E . Let $\epsilon \in F^\times$ such that $\omega_{E/F}(\epsilon) = -1$. Set

$$\begin{cases} V^+ = (V_E, N_{E/F}) \oplus \mathbb{H} \\ V^- = \epsilon V^+ \end{cases}$$

where $V_E = E$ as a 2-dimensional vector space over F . Let $V = V^+$ with a quadratic form q , we write elements of V as

$$v = \begin{pmatrix} a & x \\ \sigma(x) & b \end{pmatrix}, \text{ and the quadratic form } q(v) = N_{E/F}(x) - ab.$$

The quadratic character $\chi_V(\lambda) = (\lambda, \det V)_F = \omega_{E/F}(\lambda)$ for $\lambda \in F^\times$, and the Hasse-invariant

$$\epsilon(V) = 1.$$

For any vector $v \in V^-$, the same vector space over F with V^+ , we define a quadratic form

$$q^-(v) = \epsilon(N(x) - ab).$$

The quadratic character $\chi_{V^-} = \omega_{E/F}$ and the Hasse-invariant of V^- is -1 .

There is a $(GL_2(E) \times F^\times)$ -action on V , for $(g, \lambda) \in GL_2(E) \times F^\times$ and $v = \begin{pmatrix} a & x \\ \sigma(x) & b \end{pmatrix} \in V$, set

$$(g, \lambda).v = \lambda g v \sigma(g)^t \quad (\text{as a matrix multiplication}).$$

We have

$$q(\lambda g v \sigma(g)^t) = \lambda^2 N(\det(g)) q(v).$$

Then $(g, \lambda) \in GSO(V)$ with the similitude factor $\lambda^2 \cdot N(\det g)$. The situation is the same for $GSO(V^-)$. And by Theorem 2.1, we have an isomorphism

$$GSO(V, q) \cong \frac{GL_2(E) \times F^\times}{\Delta E^\times}, \text{ where } \Delta(t) = (t, N_{E/F}(t^{-1})), t \in E^\times.$$

Later, we denote $GSO(V^+)$ as $GSO(3, 1)$, and denote $GSO(V^-)$ as $GSO(1, 3)$.

Representations of $GSO(V)$ Assume χ is a character of F^\times . Given a representation π of $GL_2(E)$, then $\pi \boxtimes \chi$ is a representation of $GSO(V)$ if and only if $\omega_\pi = \chi \circ N_{E/F}$. Fix a parabolic subgroup P , which stabilizes the isotropic line $\left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\}$ in V , assume $P = MN$ and the Levi subgroup M is a quotient of $T(E) \times F^\times$, where $T(E)$ is the split torus of $GL_2(E)$. In fact, M is isomorphic to $F^\times \times E^\times$. Assume $\mu : E^\times \rightarrow \mathbb{C}^\times$ is a character, denote $V_E = E$ as a 2-dimensional quadratic space over F , the quadratic form q coincides with the norm map $N_{E/F}$. Consider the normalized induced representation $I_P(\chi, \mu)$ of $GSO(V)$.

Lemma 4.1. [GI11, Lemma A.6] For a character μ of E^\times , we have

$$I_P(\chi, \mu) = \pi((\chi \circ N_{E/F})\mu, \mu^\sigma) \boxtimes (\chi \cdot \mu|_{F^\times})$$

Fix a nontrivial additive character ψ of F . Let Ω_ψ be the Weil representation of

$$R = GO(V) \times GSp^+(W).$$

In fact, we shall only consider the theta correspondence for $GSO(V) \times GSp^+(W)$. There is no significant loss in restricting to $GSO(V)$ because of the following lemma.

Lemma 4.2. Let π (resp. τ) be an irreducible representation of $GSp^+(W)$ (resp. $GO(V)$) and suppose that

$$\text{Hom}_R(\Omega, \pi \otimes \tau) \neq 0.$$

Then the restriction of τ to $GSO(V)$ is irreducible. If $\nu_0 = \lambda_V^{-2} \circ \det$ is the unique nontrivial quadratic character of $GO(V)/GSO(V)$, then $\tau \otimes \nu_0$ does not participate in the theta correspondence with $GSp^+(W)$.

We follow the proof of Gan in [GT11b, Lemma 2.4].

Proof. Note that τ is irreducible when restricted to $GSO(V)$ if and only if $\tau \otimes \nu_0 \neq \tau$. Consider $\tau|_{O(V)} = \oplus_i \tau_i$ by [Rob96], and $\tau_i|_{SO(V)}$ is irreducible and $\tau_i \otimes \nu_0 \neq \tau_i$ by Rallis's result, which can be found in [Pra93, Section 5, Pg 282]. And if τ participates in the theta correspondence with $GSp^+(W)$, we know $\tau_i \otimes \nu_0$ does not participate in the theta correspondence with $Sp(W)$ by Rallis's result as well. This implies that $\tau \otimes \nu_0 \neq \tau$ and $\tau \otimes \nu_0$ does not participate in the theta correspondence with $GSp^+(W)$. \square

Proposition 4.3. Suppose that π is a supercuspidal representation of $GO(V)$ (resp. $GSp^+(W)$). Then $\Theta_\psi(\pi)$ is either zero or is an irreducible representation of $GSp^+(W)$ (resp. $GO(V)$).

One can see the proof in [Kud96] and [GT11b]. This proposition also holds when π is a discrete series [Mui08], but we do not use this result.

From now on, all representations are smooth in the remaining part of this section. If we say that χ is a character of $GL_2(F)$, it means $\chi \circ \det : GL_2(F) \rightarrow \mathbb{C}$. And a character χ of the inner form $D^\times(F)$ of $GL_2(F)$ means $\chi \circ N_{D/F}$. The unnormalized induced representation is denoted by $\text{ind}_H^G S(V)$, where H is a closed subgroup and $S(V)$ is a smooth representation of H .

4.1 Whittaker model

Given an irreducible representation Σ of $GSO(V)$, we consider the Whittaker model of the big theta lifts $\theta_\psi(\Sigma)$. Recall, for $n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in GSp^+(W)$ and $\phi \in S(NE^\times \times V)$, we have

$$n(x)\phi(t, v) = \psi(tx \cdot q(v))\phi(t, v).$$

Assume $Wh = S(NE^\times) \otimes S(V)$ is the Schwartz function space. Given $a \in F^\times$, set

$$\Omega_{N, \psi_a} = Wh / \langle n(x)\phi - \psi(ax)\phi \rangle.$$

Given an irreducible representation $\Sigma \cong \pi \boxtimes \chi$ of $GSO(V)$, consider the set

$$V_a = \{(t, v) \in NE^\times \times V \mid tq(v) = a\},$$

pick a point $y_a = (1, a) \in V_a$, where $a = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \in V$. Let Z be the center of $GL_2(W)$. Since $GSO(V) \times Z$ acts on V_a transitively with the action

$$(h, b)(t, v) = (\lambda_V(h) \cdot b^2 t, b h^{-1} v),$$

then y_a has the stabilizer

$$J_a = \{(h, b) \in (GSO(V) \times Z) \cap R_0 \mid h y_a = b y_a\} \subset P_a \times Z,$$

where P_a is the parabolic subgroup of $GSO(V)$ fixing the anisotropic line $F y_a$ in V .

Proposition 4.4. *Given an irreducible smooth representation Σ of $GSO(V)$, we have*

$$Hom_N(\Theta_\psi(\Sigma), \psi_a) \cong Hom_{GSO(V)}(\Omega_{N, \psi_a}, \Sigma) \cong Hom_{SO(y_a^\perp)}(\Sigma^\vee, \mathbb{C}).$$

Proof. We restrict the functions to the subset V_a and consider the exact sequence

$$0 \longrightarrow \ker \longrightarrow Wh \longrightarrow S(V_a) \longrightarrow 0,$$

where $V_a = \{(t, v) \in NE^\times \times V \mid tq(v) = a\}$. Since the functions ϕ lying inside \ker can be generated by the elements of form $n(x)\phi - \psi(ax)\phi$ and N acts on $S(V_a)$ as a character, then we have

$$(\ker)_{N, \psi_a} = 0, \quad \Omega_{N, \psi_a} \cong S(V_a) \cong ind_{J_a}^{GSO(V) \times Z} \mathbb{C}.$$

By easy computation, we can get

$$ind_{J_a}^{GSO(V) \times Z} \mathbb{C} = ind_{P_a \times Z}^{GSO(V) \times Z} ind_{J_a}^{P_a \times Z} \mathbb{C} \cong ind_{P_a \times Z}^{GSO(V) \times Z} S(F^\times) \cong ind_{P_a}^{GSO(V)} S(F^\times)$$

and $S(F^\times) \cong ind_{SO(y_a^\perp)}^{P_a} \mathbb{C}$, so the Jacquet module Ω_{N, ψ_a} is

$$\Omega_{N, \psi_a} \cong ind_{SO(y_a^\perp)}^{GSO(V)} \mathbb{C}.$$

Then by the universal property of theta lifting and Frobenius Reciprocity, we have

$$\begin{aligned} Hom_N(\Theta_\psi(\Sigma), \psi_a) &\cong Hom_{N \times GSO(V)}(\Omega, \psi_a \otimes \Sigma) \\ &\cong Hom_{GSO(V)}(\Omega_{N, \psi_a}, \Sigma) \\ &\cong Hom_{GSO(V)}(ind_{SO(y_a^\perp)}^{GSO(V)} \mathbb{C}, \Sigma) \\ &\cong Hom_{GSO(V)}(\Sigma^\vee, Ind_{SO(y_a^\perp)}^{GSO(V)} \mathbb{C}) \\ &\cong Hom_{SO(y_a^\perp)}(\Sigma^\vee|_{SO(y_a^\perp)}, \mathbb{C}). \end{aligned}$$

□

Since $SO(y_a^\perp) \cong SO(3)$, then Σ is $SO(3)$ -distinguished if and only if $\Theta_\psi(\Sigma)$ is (N, ψ_a) -generic.

Proposition 4.5. *If $E = F \times F$ is split, D is a quaternion algebra over F . Assume π is an irreducible smooth representation of $GSO(D(E))$, then π is $PD^\times(F)$ -distinguished if and only if $\pi = \tau \boxtimes \tau^\vee$, where τ is an irreducible representation of $D^\times(F)$.*

Mixed model Now we introduce the mixed model to deal with the theta lift from $GSO(V)$ to $GSp^+(W)$. From the Witt decomposition $W = X + Y$ and

$$V = Fv_0 + V_E + Fv_0^* \quad \left(v_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } v_0^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

we obtain

$$V \otimes X + V \otimes Y = (Fv_0 \otimes W) + V_E \otimes (X + Y) + (Fv_0^* \otimes W).$$

Note that there are two isotropic subspaces $Fv_0 \otimes X$ and $Fv_0^* \otimes Y$ paired via the natural symplectic form $\langle -, - \rangle$ on $V \otimes W$. The intertwining map

$$I : S(V \otimes Y) \rightarrow S(V_E \otimes Y) \otimes S(W \otimes v_0^*)$$

is given by a partial Fourier transform: for $v \in V_E \otimes Y$, $x \in v_0^* \otimes X$, $y \in v_0^* \otimes Y$, and $z \in v_0 \otimes Y$, we have

$$(I\varphi)(v, x, y) = \int_F \psi(zx) \varphi\left(\begin{pmatrix} z \\ v \\ y \end{pmatrix}\right) dz$$

Let Q be the maximal parabolic subgroup of $SO(V)$ which stabilizes Fv_0 , and let U be its unipotent radical. Then for $h = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in U$, we have

$$I(\omega_\psi(h)\varphi)(v, x, y) = \psi(x \cdot \text{tr}_{E/F}(b\sigma(v))) \psi(-N(b)xy) (I\varphi)(v, x, y).$$

For an element m in the Levi subgroup of Q , set $m = \left(\begin{pmatrix} 1 & \\ & d \end{pmatrix}, \lambda \right)$, we have

$$I(\omega_\psi(m)\varphi)(v, x, y) = |\lambda| (I\varphi)\left(\frac{v}{\lambda\sigma(d)}, \frac{x}{\lambda N(d)}, \lambda y\right).$$

For $g \in Sp(W)$, regard $(I\varphi)(v, x, y) = (I\varphi)(x, y)(v)$ as a function defined on $S(v_0^* \otimes W)$ taking values in $S(V_E \otimes X)$, then we have

$$I(\omega_\psi(g)\varphi)(v, x, y) = \left(\omega_0(g) (I\varphi)\left(g^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) \right)(v)$$

where ω_0 is the Weil representation of $Sp(W) \times O(V_E)$.

Now we extend the Weil representation ω_ψ on $SO(V) \times Sp(W)$ to the group R_0 by

$$\omega_\psi(g, h)\phi = |\lambda_V(h)|^{-\frac{1}{8} \dim V \cdot \dim W} \omega(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda_W(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in Sp(W).$$

And consider the compact induction

$$\Omega = \text{ind}_{R_0}^R \omega_\psi.$$

Assume ψ_E is an additive character of E , which is isomorphic to the unipotent radical subgroup U of $GSO(V)$, defined by $\psi_E(b) = \psi(\text{tr}_{E/F}(b))$.

Proposition 4.6. *Assume σ is an irreducible representation of $GS p^+(W)$, then*

$$Hom_U(\Theta_\psi(\sigma), \psi_E) \cong Hom_N(\sigma, \psi).$$

Proof. Set $Wh = S(NE^\times) \otimes S(W) \otimes S(V_E)$, consider the short exact sequence with respect to the delta function taking value at the point $0 \in W$, then we can get

$$0 \longrightarrow \ker \longrightarrow Wh \longrightarrow S(NE^\times) \otimes S(V_E) \longrightarrow 0.$$

Since U acts trivially on $S(NE^\times) \otimes S(V_E)$, then

$$\Omega_{U, \psi_E} = Wh / \langle n(b)\phi - \psi_E(b)\phi \rangle \cong (\ker)_{U, \psi_E}.$$

The group $GS p^+(W) \times U$ acts transitively on the set $NE^\times \times (W - \{0\})$, pick a point $(1, 1, 0)$ with stabilizer $N \times U$. Then $\ker \cong ind_{N \times U}^{GS p^+(W) \times U} S(V_E)$. Here $S(V_E)$ is a $N \times U$ -module, for $f \in S(V_E)$, there is a Schwartz function $\bar{f} \in Wh$ such that

$$f(v) = \bar{f}(1, 1, 0)(v).$$

For $(n, b) \in N \times U$, we have

$$\omega_\psi(n, b)f(v) = \psi_E(b\sigma(v))\psi(nN(v)) \cdot f(v) \text{ for } n = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}.$$

Consider $(\ker)_{U, \psi_E}$, then only the function defined at point $v = 1$ can survive, which means

$$(\ker)_{U, \psi_E} \cong ind_N^{GS p^+(W)} \mathbb{C}_\psi.$$

Hence, we have

$$\begin{aligned} Hom_U(\Theta_\psi(\sigma), \psi_E) &\cong Hom_{U \times GS p^+(W)}(\Omega, \psi_E \otimes \sigma) \\ &\cong Hom_{GS p^+(W)}(\Omega_{U, \psi_E}, \sigma) \\ &\cong Hom_{GS p^+(W)}((\ker)_{U, \psi_E}, \sigma) \\ &\cong Hom_{GS p^+(W)}(ind_N^{GS p^+(W)} \psi, \sigma) \\ &\cong Hom_N(\sigma^\vee, \psi^{-1}) \\ &\cong Hom_N(\sigma, \psi). \end{aligned}$$

□

Corollary 4.7. *Assume σ is an irreducible representation of $GS p^+(W)$, then σ is ψ -generic if and only if the big theta lift $\Theta_\psi(\sigma)$ is generic.*

4.2 Principal series

Assume B is a Borel subgroup of $GS p(W)$ with a Levi subgroup T . We define a subgroup $B^+ = B \cap GS p^+(W)$ and its torus subgroup $T^+ = T \cap B^+$. Then we can begin to compute the local theta lift from $GS p^+(W)$ to $GSO(V)$.

Let us start from the irreducible principal series representations of $GS p(W)$. Set

$$GL_2^+ = GL_2^+(F) = GS p^+(W) = \{g \in GL_2(F) | \omega_{E/F}(g) = 1\}.$$

Lemma 4.8. *Assume τ is an irreducible infinitely dimensional representation of $GSp(W)$, then $\tau|_{GL_2^+}$ is reducible if and only if $\tau \otimes \omega_{E/F} \cong \tau$, in which case, we call it dihedral with respect to E .*

Proof. Let $\omega_{E/F} : GSp(W) \rightarrow \mathbb{C}^\times$ be a character of $GSp(W)$, set $\omega_{E/F}(g) = \omega(\det(g))$, then

$$\ker \omega_{E/F} = GL_2^+.$$

And

$$Hom_{GL_2^+}(\tau|_{GL_2^+}, \tau|_{GL_2^+}) \cong Hom_{GL_2(F)}(\tau, Ind(\tau|_{GL_2^+})) \cong Hom_{GL_2(F)}(\tau, \tau \oplus (\tau \otimes \omega_{E/F})).$$

Its dimension is two if and only if $\tau \cong \tau \otimes \omega_{E/F}$. \square

Theorem 4.9. *Assume $\tau = \pi(\chi_1, \chi_2)$ is an irreducible principal series representation of $GSp(W)$. Then $\chi_1 \neq \chi_2 \cdot |\cdot|^{\pm 1}$. Denote the norm map $N_{E/F}$ by N .*

(i) *If $\tau \not\cong \tau \otimes \omega_{E/F}$ and $\chi_1 \neq \chi_2 \omega_{E/F} |\cdot|^{\pm 1}$. Then $\tau|_{GL_2^+}$ is irreducible and*

$$\Theta_\psi(\tau^+) = \theta_\psi(\tau^+) \cong \pi(\chi_2 \circ N, \chi_1 \circ N) \boxtimes \chi_1 \chi_2 \chi_V.$$

(ii) *If $\tau = \pi(\chi_3, \chi_3 \omega_{E/F})$ is dihedral with respect to E , then $\tau|_{GL_2^+} \cong \tau^+ \oplus \tau^-$ and*

$$\Theta_\psi(\tau^+) = \theta_\psi(\tau^+) \cong \pi(\chi_3 \circ N, \chi_3 \circ N) \boxtimes \omega_{\tau^+ \omega_{E/F}}, \text{ while the other one } \theta_\psi(\tau^-) = 0,$$

where ω_{τ^+} is the central character of the representation τ^+ .

(iii) *If $\tau = \pi(|\cdot|^{1/2}, |\cdot|^{-1/2} \omega_{E/F}) \otimes \chi_4$ or $\pi(|\cdot|^{-1/2}, |\cdot|^{1/2} \omega_{E/F}) \otimes \chi_4$, then $\tau|_{GL_2^+}$ is irreducible and*

$$\Theta_\psi(\tau|_{GL_2^+}) \cong (\pi(|\cdot|_E^{1/2}, |\cdot|_E^{-1/2}) \otimes \chi_4 \circ N) \boxtimes \chi_4 \circ N \text{ and } \theta_\psi(\tau|_{GL_2^+}) \cong \chi_4 \circ N \boxtimes \chi_4 \circ N.$$

Proof. (i) The strategy is to compute the Jacquet module. Let N be the unipotent radical of the group $GSp^+(W)$. Set

$$J_N = S(V \times NE^\times) / \langle n(x)\phi - \phi \rangle, \text{ where } n(x) \in N.$$

Consider the restriction map $Wh = S(NE^\times \times V) \rightarrow S(NE^\times \times \{0\})$, i.e. taking values at $0 \in V$, then we have an exact sequence of R -modules

$$0 \longrightarrow S(NE^\times) \otimes S(V - \{0\}) \longrightarrow Wh \longrightarrow S(NE^\times) \longrightarrow 0.$$

Recall $n(x)\phi(t, v) = \psi(tx \cdot q(v))\phi(t, v)$ and

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \phi(t, v) = \chi_V(b^{-1})|b|^{-2}\phi(abt, b^{-1}v),$$

where $ab \in NE^\times$. Then N acts trivially on $S(NE^\times)$. Since the Jacquet functor is exact, we obtain an exact sequence of $(T^+ \times GSO(V))$ -modules

$$0 \longrightarrow R_{B^+}(S(NE^\times) \otimes S(V - \{0\})) \longrightarrow J_N \longrightarrow S(NE^\times) \longrightarrow 0, \quad (*)$$

where $B^+ \subset GL_2^+$ is the Borel subgroup, T^+ is the diagonal torus subgroup. Observe that the set $NE^\times = T^+ \cdot 1_F$ with stabilizer $T^+ \cap SL_2$, then

$$S(NE^\times) \cong \text{ind}_{T^+ \cap SL_2(F)}^{T^+} \mu,$$

where $\mu\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) = \chi_V(b^{-1})|b|^{-2}$ if $\begin{pmatrix} a & \\ & b \end{pmatrix} \in SL_2(F)$.

Now we consider another restriction map $S(NE^\times \times (V - \{0\})) \rightarrow S(NE^\times \times (V_0 - \{0\}))$, where $V_0 = \{v \in V \mid q(v) = 0\}$, then there is an exact sequence

$$0 \longrightarrow S(NE^\times \times (V - V_0)) \longrightarrow S(NE^\times \times (V - \{0\})) \longrightarrow S(NE^\times \times (V_0 - \{0\})) \longrightarrow 0. \quad (**)$$

For a function $\phi \in S(NE^\times \times (V_0 - \{0\}))$, we have $n(x)\phi(t, v) = \phi(t, v)$ which is a trivial action. For the function $\phi \in S(NE^\times \times (V - V_0))$, we have

$$\int_{\mathfrak{p}^{-k}} n(x)\phi(t, v)dx = \int_{\mathfrak{p}^{-k}} \psi(xt \cdot q(v))\phi(t, v)dx = 0$$

for sufficiently large $k \in \mathbb{Z}$. The reason is that for $q(v) \neq 0$, we assume the conduct of ψ is \mathfrak{p}^{-k} , then the integral of a non-trivial additive character ψ on the conduct is zero. Since the function $\phi \in S(NE^\times \times (V - V_0))$ can be generated by the elements of forms $n(x)\phi - \phi$, by taking Jacquet functor R_{B^+} of $(**)$, we can get an isomorphism of T^+ -modules

$$R_{B^+}(S(NE^\times) \otimes S(V - \{0\})) \cong S(NE^\times \times (V_0 - \{0\})).$$

Pick a point $v = \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \in NE^\times \times (V_0 - \{0\})$, the similitude group $GSO(V)$ acts transitively on the set $NE^\times \times (V_0 - \{0\})$, then we have $GSO(V).v = NE^\times \times (V_0 - \{0\})$ as sets and the stabilizer of v is the derived subgroup $[P, P]$, where $P \subset GSO(3, 1)$ is the stabilizer of the variety $NE^\times \times F^\times = \left\{(t, x) \in NE^\times \times (V_0 - \{0\}) \mid x = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in V_0 - \{0\}\right\}$. Hence, we have

$$S(NE^\times \times (V_0 - \{0\})) \cong \text{ind}_P^{GSO(3,1)} \text{ind}_{[P,P]}^P \mathbb{C} = \text{ind}_P^{GSO(V)} S(F^\times \times NE^\times),$$

i.e. $R_{B^+}(S(NE^\times) \otimes S(V - \{0\})) \cong \text{ind}_P^{GSO(3,1)} S(NE^\times \times F^\times)$.

Now we can rephrase the exact sequence $(*)$ as follow:

$$0 \longrightarrow \text{ind}_P^{GSO(V)} S(NE^\times \times F^\times) \longrightarrow J_N \longrightarrow \text{ind}_{T^+ \cap SL_2(F)}^{T^+} \mu \longrightarrow 0.$$

Let $\chi = \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}$ be a character of T^+ . Since the functor $\text{Hom}_{T^+}(-, \chi)$ is left exact and contravariant, we have a long exact sequence

$$0 \longrightarrow \text{Hom}_{T^+}(\text{ind}_{T^+ \cap SL_2(F)}^{T^+} \mu, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \longrightarrow \text{Hom}_{T^+}(J_N, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \longrightarrow$$

$$\text{Hom}_{T^+}(\text{ind}_P^{GSO(3,1)} S(NE^\times \times F^\times), \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \rightarrow \text{Ext}_{T^+}^1(\text{ind}_{T^+ \cap SL_2(F)}^{T^+} \mu, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \rightarrow \dots$$

- the first term

$$\begin{aligned} Hom_{T^+}(ind_{T^+ \cap SL_2(F)}^{T^+} \mu, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) &\cong Hom_{T^+}(\delta^{-1/2}(\chi_1^{-1} \times \chi_2^{-1})|_{T^+}, Ind_{T^+ \cap SL_2(F)}^{T^+} \mu^{-1}) \\ &\cong Hom_{T^+ \cap SL_2(F)}(\delta^{-1/2}(\chi_1^{-1} \times \chi_2^{-1})|_{T^+ \cap SL_2(F)}, \mu^{-1}) \\ &= 0. \end{aligned}$$

The last identity holds because for $ab = 1$, we have

$$\delta^{-1/2}(\chi_1^{-1} \times \chi_2^{-1}) \begin{pmatrix} a & \\ & b \end{pmatrix} = |b| \cdot \frac{\chi_1}{\chi_2}(b) \neq \mu^{-1}(b) = \chi_V(b)|b|^2 \quad \text{unless } \chi_1 = \chi_V \chi_2.$$

- the fourth term

$$\begin{aligned} Ext_{T^+}^1(ind_{T^+ \cap SL_2(F)}^{T^+} \mu, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) &\cong Ext_{T^+}^1(\delta^{-1/2}(\chi_1^{-1} \times \chi_2^{-1})|_{T^+}, Ind_{T^+ \cap SL_2(F)}^{T^+} \mu^{-1}) \\ &= 0 \end{aligned}$$

by the same trick if $\chi_1 \neq \chi_2 \chi_V| - |$.

- Hence, for $\chi_1 \neq \chi_2 \chi_V| - |$, we have

$$\begin{aligned} Hom(\Theta_\psi(\tau|_{GL_2^+}), \mathbb{C}) &\cong Hom_{GSp^+(W)}(\Omega, \tau|_{GL_2^+}) \\ &\cong Hom_{T^+}(J_N, \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \\ &\cong Hom_{T^+}(ind_P^{GSO(3,1)} S(NE^\times \times F^\times), \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}). \end{aligned}$$

Recall for $ab \in NE^\times$, we have

$$\left(\begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} 1 & \\ & d \end{pmatrix}, \lambda \right) \phi(t, x) = \chi_V(b^{-1})|b|^{-2}|\lambda^2 N(d)|^{-1} \phi(ab\lambda^2 N(d)t, \lambda^{-1}b^{-1}x).$$

Due to [GG05, Lemma (9.4)], the χ -isotropic quotient of $ind_P^{GSO(V)} S(F^\times \times NE^\times)$ is $ind_P^{GSO(V)} \chi^{-1}$. Consider the twisted version, then we have

$$Hom_{T^+}(ind_P^{GSO(V)} S(NE^\times \times F^\times), \delta^{1/2}(\chi_1 \times \chi_2)|_{T^+}) \cong Hom(IP(\chi_1 \chi_2 \chi_V, \chi_1 \circ N), \mathbb{C}),$$

where $P = MN$ and $M \cong GL_1(F) \times E^\times$. Hence

$$Hom(\Theta_\psi(\tau|_{GL_2^+}), \mathbb{C}) \cong Hom(\pi(\chi_2 \circ N, \chi_1 \circ N) \boxtimes \chi_1 \chi_2 \chi_V, \mathbb{C}),$$

and taking the smooth part, we can get

$$\Theta_\psi(\tau|_{GL_2^+}) \cong \pi(\chi_2 \circ N, \chi_1 \circ N) \boxtimes (\chi_1 \chi_2 \chi_V) \text{ if } \chi_1 \neq \omega_{E/F} \chi_2| - |^{\pm 1}.$$

- (ii) By the computation above, we have

$$Hom_{GSp^+(W)}(\Omega, \tau|_{GL_2^+}) \cong Hom_{T^+}(ind_P^{GSO(3,1)} S(NE^\times \times F^\times), \delta^{1/2}(\chi_3 \times \chi_3 \omega_{E/F})|_{T^+}).$$

Then $\Theta_\psi(\tau^+) \oplus \Theta_\psi(\tau^-) \cong \pi(\chi_3 \circ N, \chi_3 \circ N) \boxtimes \chi_3^2$ which is irreducible. Hence, only one of the two representations τ^\pm participates in the theta correspondence between $GSO(V)$ and GL_2^+ , denoted by τ^+ . And $\theta_\psi(\tau^-) = 0$. Since $\Theta_\psi(\tau^+)$ is generic, we get that τ^+ is ψ -generic by Corollary 4.7.

(iii) We may assume χ_4 is trivial, then $\tau|_{GL_2^+} \cong \pi(|-|^{-1/2} \cdot \omega_{E/F}, |-|^{1/2})|_{GL_2^+}$ in both cases. Then using the same trick, we can obtain $\Theta_\psi(\tau|_{GL_2^+}) \cong \pi(|-|_E^{1/2}, |-|_E^{-1/2}) \boxtimes \mathbb{C}$, where $|-|_E = |-| \circ N_{E/F}$. And the small theta lifting is $\theta_\psi(\tau|_{GL_2^+}) \cong \mathbb{C} \boxtimes \mathbb{C}$. In general, we have

$$\Theta_\psi(\tau|_{GL_2^+}) \cong (\pi(|-|_E^{1/2}, |-|_E^{-1/2}) \otimes \chi_4 \circ N) \boxtimes \chi_4 \circ N \text{ and } \theta_\psi(\tau|_{GL_2^+}) \cong \chi_4 \circ N \boxtimes \chi_4 \circ N.$$

□

Remark 4.10. *The method taking the long exact sequence is also suitable for reducible principal series representation τ of $GL_2(F)$.*

Let us turn the tables around. Assume $\Sigma = \pi \boxtimes \chi$ is an irreducible representation of $GSO(V)$, then π must be a Base change representation coming from $GL_2(F)$ if $\theta_\psi(\Sigma) \neq 0$. Otherwise, if π is not a Base change representation and $\tau = \theta_\psi(\Sigma)$ is non-vanishing, then there exists a nonzero map from Ω to $\theta_\psi(\Sigma) \otimes \Sigma = \tau \otimes \theta_\psi(\tau)$. We will see that all $\theta_\psi(\tau) = \Sigma$ has the form $BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$, i.e. π is a base change representation of $GL_2(E)$, which is a contradiction. Let us just consider the big theta lift from $GSO(V)$ to $GSp^+(W)$ due to the Howe duality.

Theorem 4.11. *Assume π is an irreducible principal series representation of $GL_2(E)$, and π is a base change of a representation τ of $GL_2(F)$.*

(i) *If $\tau = \pi(\chi_1, \chi_2)$ is a principal series and not dihedral with respect to E , then*

$$(a) \quad \Theta_\psi(\pi \boxtimes \chi_1 \chi_2 \omega_{E/F}) = \pi(\chi_2, \chi_1)|_{GL_2^+} \text{ and}$$

$$(b) \quad \Theta_\psi(\pi \boxtimes \chi_1 \chi_2) = \pi(\chi_2, \chi_1 \omega_{E/F})|_{GL_2^+}.$$

(ii) *If τ is a dihedral principal series, i.e. $\pi = \pi(\chi_3 \circ N, \chi_3 \circ N)$, where χ_3 is a character of F^\times , we consider the representations $\Sigma_1 = \pi \boxtimes \chi_3^2 \omega_{E/F}$ and $\Sigma_2 = \pi \boxtimes \chi_3^2$, then*

$$(a) \quad \Theta_\psi(\Sigma_1) = \pi(\chi_3, \chi_3)|_{GL_2^+} \text{ and}$$

$$(b) \quad \Theta_\psi(\Sigma_2) = \tau^+, \text{ where } \tau^+ \text{ is the } \psi\text{-generic component of } \pi(\chi_3, \chi_3 \omega_{E/F})|_{GL_2^+}.$$

(iii) *If τ is dihedral supercuspidal with respect to E , i.e. $\pi = BC(\tau) = \pi(\chi_4, \chi_4^\sigma)$ and $\chi_4 \neq \chi_4^\sigma$, then the theta lift $\theta_\psi(\pi \boxtimes \omega_\tau)$ is zero and*

$$\Theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) = \theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) = \tau^+,$$

where τ^+ is the ψ -generic component of $\tau|_{GL_2^+}$.

Proof. (i) Consider the mixed model, set $Wh = S(NE^\times) \otimes S(W) \otimes S(V_E)$. Fix the parabolic subgroup $P \subset GSO(V)$ which stabilizes the line Fv_0 , and

$$P = MU, \text{ where } U = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} : b \in E \right\} \cong E.$$

Consider the restriction map, i.e. taking the value at $0 \in W$, we get an exact sequence

$$0 \longrightarrow \ker \longrightarrow Wh \longrightarrow S(NE^\times) \otimes S(V_E) \longrightarrow 0 \quad (*)$$

as $GSp^+(W) \times GSO(V)$ -modules. Set $J_U = Wh / \langle n(b)\phi - \phi \rangle$. The unipotent subgroup U acts trivially on $S(NE^\times) \times S(V_E)$. And the group $GSp^+(W) \times M$ acts transitively on the set $NE^\times \times (W - \{0\})$. The Jacquet module of \ker is isomorphic to

$$\text{ind}_Q^{GSp^+(W) \times M} \mathbb{C} \cong \text{ind}_{B^+ \times M}^{GSp^+(W) \times M} S(NE^\times \times F^\times),$$

where

$$Q = \left\{ (g, m) \in (GSp^+(W) \times M) \cap R_0 \mid g_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} s \cdot N_{E/F}(a) \\ 0 \end{pmatrix} \right\} \subset B^+ \times M,$$

is the stabilizer of $(1, 1, 0) \in NE^\times \times (W - \{0\})$, and

$$m = \left\{ \begin{pmatrix} a \\ d \end{pmatrix}, s \right\} \quad \text{and} \quad g_1 = g \begin{pmatrix} \lambda_W(g^{-1}) & \\ & 1 \end{pmatrix}.$$

Recall, for $(g, m) \in T^+ \times M$ and $f \in S(NE^\times) \otimes S(F^\times)$, we have

$$(g, m)f(t, x, 0)(0) = |s \cdot N(d)|^{-1} \left(\omega_0(g_1) f \left(\det g \cdot \lambda_V(m) \cdot t, g^{-1} \begin{pmatrix} s \cdot N(a)x \\ 0 \end{pmatrix} \right) \right)(0).$$

Taking the Jacquet functor on the exact sequence $(*)$, we obtain

$$0 \longrightarrow \text{ind}_{B^+ \times M}^{GSp^+(W) \times M} S(NE^\times \times F^\times) \longrightarrow J_U \longrightarrow S(NE^\times) \otimes S(V_E) \longrightarrow 0,$$

which is an exact sequence as $GSp^+(W) \times M$ -modules. By the similar computation in Theorem 4.9, taking the long exact sequence, we can get

$$\text{Hom}_{GSO(V)}(\Omega, \pi \boxtimes (\chi_1 \chi_2 \omega_{E/F})) \cong \text{Hom}(\pi(\chi_2, \chi_1)|_{GL_2^+}, \mathbb{C}) \text{ if } \chi_1 \neq \chi_2 \omega_{E/F}.$$

Since τ is not dihedral, then we take the smooth part to get

$$\Theta_\psi(\pi \boxtimes \chi_1 \chi_2 \omega_{E/F}) = \pi(\chi_2, \chi_1)|_{GL_2^+}.$$

Since $BC(\tau) = BC(\pi(\chi_1 \omega_{E/F}, \chi_2))$, we can easily get

$$\Theta_\psi(\pi \boxtimes \chi_1 \chi_2) = \pi(\chi_2, \chi_1 \omega_{E/F})|_{GL_2^+},$$

which is an irreducible representation of $GSp^+(W)$.

- (ii) If τ is dihedral with respect to E , and $\pi = BC(\pi(\chi_3, \chi_3))$, then $\Theta_\psi(\Sigma_1) = \pi(\chi_3, \chi_3)|_{GL_2^+}$. By comparing with Theorem 4.9(ii), we can obtain $\theta_\psi(\Sigma_2) = \tau^+$, where τ^+ is the ψ -generic component of $\pi(\chi_3, \chi_3 \omega_{E/F})$. By comparing with the representations of GL_2^+ , we can get

$$\Theta_\psi(\Sigma_2) = \theta_\psi(\Sigma_2) = \tau^+.$$

- (iii) If $\pi = \pi(\chi_4, \chi_4^\sigma)$ and $\chi_4 \neq \chi_4^\sigma$, we will use Local-Global principle in Theorem 4.17 to show that

$$\theta_\psi(\tau^+) = \pi \boxtimes \omega_{\tau^+} \omega_{E/F}, \text{ where } \tau^+ \text{ is } \psi\text{-generic component of } \tau|_{GL_2^+}.$$

By the Howe duality for the similitude groups, we can obtain

$$\Theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) = \theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) \cong \tau^+.$$

By comparing with the theta lifts from $GSp^+(W)$ to $GSO(V)$, we have $\theta_\psi(\pi \boxtimes \omega_\tau) = 0$. \square

4.3 Steinberg Representations

Now we consider the Steinberg representation which is the unique quotient of a reducible principal series representation.

Theorem 4.12. *Let St_F be the Steinberg representation of $GL_2(F)$, then $St = St_F|_{GL_2^+(F)}$ is irreducible and $\Theta(St) = \theta(St) \cong BC(St_F) \boxtimes \omega_{E/F}$.*

Proof. Set $J_N = S(NE^\times \times V)/\langle n(x) \cdot \phi - \phi \rangle$ for $\phi \in S(NE^\times \times V)$. As in the proof of Theorem 4.9, we can get the exact sequence

$$0 \longrightarrow \text{ind}_P^{GSO(3,1)} S(NE^\times \times F^\times) \longrightarrow J_N \longrightarrow \text{ind}_{T^+ \cap SL_2(F)}^{T^+} \mu \longrightarrow 0.$$

where $\mu\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) = \chi_V(b^{-1})|b|^{-2}$, for $ab = 1$. By the definition of Steinberg representation, one has a short exact sequence

$$0 \longrightarrow St_F \longrightarrow \text{Ind}_B^{GL_2(F)} \delta_B^{1/2}(|-|^{1/2} \times |-|^{-1/2}) \longrightarrow \mathbb{C} \longrightarrow 0.$$

The sequence is the same when restricted to the subgroup GL_2^+ . Since $\text{Hom}_{GL_2^+(F)}(\Omega, -)$ is a left exact functor, then we have an embedding

$$\text{Hom}_{GSp^+(W)}(\Omega, St) \hookrightarrow \text{Hom}_{GSp^+(W)}(\Omega, \text{Ind}_B^{GL_2(F)} \delta_B^{1/2}(|-|^{1/2} \times |-|^{-1/2})).$$

Since the Jacquet functor is adjoint to induction functor and the similar computation as in the proof of Theorem 4.9, we have an injection

$$(\Theta_\psi(St))^* \hookrightarrow \text{Hom}_{T^+}(J_N, \delta^{1/2}(|-|^{1/2} \times |-|^{-1/2}))$$

i.e. $\Theta_\psi(St)^* \hookrightarrow \text{Hom}(\text{ind}_{B(E)}^{GL_2(E)} \delta_B^{1/2}(|-|_E^{-1/2} \times |-|_E^{1/2}) \boxtimes \omega_{E/F}, \mathbb{C})$. Then the map

$$\text{ind}_{B(E)}^{GL_2(E)} \delta_B^{1/2}(|-|_E^{-1/2} \times |-|_E^{1/2}) \boxtimes \omega_{E/F} \rightarrow \Theta_\psi(St) \text{ is surjective.}$$

Moreover, there is a nonzero map $\Omega \rightarrow (St_E \boxtimes \omega_{E/F}) \otimes \text{Ind}_B^{GL_2(F)} \delta_B^{1/2}(|-|^{1/2} \times |-|^{-1/2})$, i.e.

$$\Theta_\psi(St_E \boxtimes \omega_{E/F}) \neq 0.$$

This is because $\text{Hom}_{GL_2^+}(\Omega_\psi, \pi(|-|^{1/2}, |-|^{-1/2})) \cong (\pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F})^*$ (the full dual), so that

$$\text{Hom}_{GSO(V)}((St_E \boxtimes \omega_{E/F})^\vee, \text{Hom}_{GL_2^+}(\Omega_\psi, \pi(\|^{1/2}, \|-^{1/2}))) \text{ is nonzero.}$$

Here $(St_E \boxtimes \omega_{E/F})^\vee$ is the smooth part of the full dual space $(St_E \boxtimes \omega_{E/F})^*$, which is a subspace of $(\pi(\|_E^{-1/2}, \|-^{1/2}) \boxtimes \omega_{E/F})^*$. This means that there is a nonzero $GSO(V) \times GL_2^+$ -equivariant map from Ω_ψ to $((St_E \boxtimes \omega_{E/F})^\vee)^* \otimes \pi(\|^{1/2}, \|-^{1/2})$. But Ω_ψ is smooth, the image is smooth as well, which lies inside

$$(St_E \boxtimes \omega_{E/F})^{\vee\vee} \otimes \pi(|-|^{1/2}, |-|^{-1/2}) \cong St_E \boxtimes \omega_{E/F} \otimes \pi(|-|^{1/2}, |-|^{-1/2}).$$

Later, we will see $\Theta_\psi(St) \neq 0$, then $\theta_\psi(St) \cong St_E \boxtimes \omega_{E/F}$, see Remark 4.15.

Now let us focus on $\Theta_\psi(St)$, which is a nonzero quotient of $\pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F}$. There are two possibilities: $\Theta_\psi(St) \cong \theta_\psi(St)$ or $\pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F}$. If $\Theta_\psi(St)$ is reducible, then there is a surjective map

$$\Omega_\psi \rightarrow St \otimes \Theta_\psi(St) = St \otimes \pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F}.$$

And there is an exact sequence

$$0 \longrightarrow St \longrightarrow Hom_{GSO(V)}(\Omega_\psi, \pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F})$$

Consider the mixed model, we can show that

$$Hom_{GSO(V)}(\Omega_\psi, \pi(|-|_E^{-1/2}, |-|_E^{1/2}) \boxtimes \omega_{E/F}) \cong (\pi(|-|_E^{1/2}, |-|_E^{-1/2}))^*,$$

which only contains a one-dimensional subrepresentation of $GL_2^+(W)$. Then we obtain that St is one-dimensional, and get a contradiction! Hence $\Theta_\psi(St) \cong \theta_\psi(St)$ is irreducible. \square

Remark 4.13. From [Kud96], the first occurrence index of St_F in the Witt towers with nontrivial character χ_V are both 4, i.e. $\Theta_\psi(St) \neq 0$, which we will reprove, then we can also get

$$\theta_\psi(St) \cong BC(St_F) \boxtimes \omega_{E/F}.$$

Theorem 4.14. Let $St_\chi = St_F \otimes \chi$ be the twisted Steinberg representation of $GL_2(F)$. Then $BC(St_\chi)$ is an irreducible representation of $GL_2(E)$, and $BC(St_\chi) \boxtimes \chi^2 \omega_{E/F}$ is an irreducible representation of $GSO(V)$. Moreover, we have

$$\theta_\psi(BC(St_\chi) \boxtimes \chi^2) = 0 \text{ and } \Theta_\psi(BC(St_\chi) \boxtimes \chi^2 \omega_{E/F}) = \theta_\psi(BC(St_\chi) \boxtimes \chi^2 \omega_{E/F}) \cong St_\chi|_{GL_2^+(F)}.$$

Proof. We may assume χ is trivial. By the definition of St_F , we have an exact sequence

$$0 \longrightarrow St_F \longrightarrow Ind_{B(F)}^{GL_2(F)} \delta_B^{1/2}(|-|^{1/2} \times |-|^{-1/2}) \longrightarrow \mathbb{C} \longrightarrow 0.$$

Now we denote $St = St_F|_{GL_2^+}$ and $St_E = BC(St_F)$. Then we have an embedding

$$0 \rightarrow Hom(\Theta(St_E \boxtimes \omega_{E/F}), \mathbb{C}) \rightarrow Hom_{GSO(V)}(\Omega, Ind_P^{GSO(V)}(\delta_B^{1/2}(|-|^{1/2} \times |-|^{-1/2}) \circ N \boxtimes \omega_{E/F})).$$

Consider the mixed model as in the proof of Theorem 4.11, we have

$$Hom_{GSO(V)}(\Omega, \pi(|-|_E^{\frac{1}{2}}, |-|_E^{-\frac{1}{2}}) \boxtimes \omega_{E/F}) \cong Hom(Ind_{B^+(F)}^{GL_2^+(F)} \delta_B^{1/2}(|-|^{-1/2} \times |-|^{1/2}), \mathbb{C}).$$

Hence, we obtain a surjection

$$Ind_{B^+(F)}^{GL_2^+(F)} \delta_B^{1/2}(|-|^{-1/2} \times |-|^{1/2}) \longrightarrow \Theta(BC(St_F) \boxtimes \omega_{E/F}) \longrightarrow 0.$$

We have shown that $\Theta_\psi(St_E \boxtimes \omega_{E/F})$ is nonzero in Theorem 4.12. Hence,

$$\theta_\psi(BC(St_F) \boxtimes \omega_{E/F}) \cong St = St_F|_{GL_2^+}.$$

With the same trick used in the proof of Theorem 4.12, we can obtain

$$\Theta_\psi(St_E \boxtimes \omega_{E/F}) = \theta_\psi(St_E \boxtimes \omega_{E/F}) = St_F|_{GL_2^+}.$$

By comparing with the theta lift from $GSp^+(W)$ to $GSO(V)$, we can get

$$\theta_\psi(BC(St_\chi) \boxtimes \chi^2) = 0.$$

□

Remark 4.15. *From the proof, we get a nonzero R -equivalent map*

$$\Omega_\psi \rightarrow St \otimes \left(Ind_P^{GSO(V)} (\delta_B^{1/2} (|-|^{-1/2} \times |-|^{1/2})) \circ N \boxtimes \omega_{E/F} \right).$$

This means $\Theta_\psi(St) \neq 0$.

In summary, we have $\Theta_\psi(St) = \theta_\psi(St) = St_E \boxtimes \omega_{E/F}$ and

$$\Theta_\psi(St_E \boxtimes \omega_{E/F}) = \theta_\psi(St_E \boxtimes \omega_{E/F}) = St_F|_{GL_2^+}.$$

4.4 Supercuspidal representations

The following proposition is well known, refer to [GI11], [Kud96].

Assume E/F is a quadratic field extension, and set $V_E = E$ to be the quadratic vector space over F and quadratic form coincides with the norm map $N_{E/F}$. Then the similitude group $GO(V_E)$ is isomorphic to $E^\times \rtimes Gal(E/F)$.

Proposition 4.16. *Let μ be an irreducible representation of E^\times , if μ is Galois invariant, then μ has two extentions μ^\pm to $GO(V_E)$, in which case, only one of them has a nonzero theta lifting to GL_2^+ , denoted by μ^+ . If μ is not Galois invariant, then $\mu^+ = ind_{GSO(V_E)}^{GO(V_E)} \mu$, and $\Theta(\mu^+)$ is a non-zero irreducible supercuspidal representation of $GL_2^+(F)$. And $ind_{GL_2^+}^{GL_2}(\Theta(\mu^+))$ is irreducible supercuspidal, which is dihedral with respect to E .*

If $\mu = \mu_F \circ N_{E/F}$ for some μ_F , then $ind_{GL_2^+}^{GL_2}(\Theta(\mu^+)) = \pi(\mu_F, \mu_F \omega_{E/F})$. Moreover, we have $\Theta(\mu^-) = 0$. The theta lifting from the character μ of E^\times to $GSp^+(W)$ is related to the automorphic induction cuspidal representation of $GSp(W)$.

Theorem 4.17. *Assume τ is a supercuspidal representation of $GL_2(F)$.*

- (i) *If τ is not dihedral with respect to E , i.e. $\tau|_{GL_2^+}$ is irreducible, then $BC(\tau)$ is a supercuspidal representation of $GL_2(E)$, and*

$$\Theta_\psi(\tau|_{GL_2^+}) = \theta_\psi(\tau|_{GL_2^+}) \cong BC(\tau) \boxtimes \omega_{E/F} \omega_\tau.$$

- (ii) *If $\tau \cong \tau \otimes \omega_{E/F}$ is dihedral with respect to E and $\phi_\tau = Ind_{WD_E}^{WD_F} \chi$, then $BC(\tau) = \pi(\chi, \chi^\sigma)$ is a principal series of $GL_2(E)$, where σ is the nontrivial element in $Gal(E/F)$, and $\chi \neq \chi^\sigma$. Let $\tau|_{GL_2^+} \cong \tau^+ \oplus \tau^-$, where τ^+ is ψ -generic, then*

$$\Theta_\psi(\tau^+) = \theta_\psi(\tau^+) \cong \pi(\chi, \chi^\sigma) \boxtimes \omega_{\tau^+} \omega_{E/F} \quad \text{and} \quad \theta_\psi(\tau^-) = 0.$$

(iii) If $\pi = BC(\tau)$ is a supercuspidal representation of $GL_2(E)$, then $\pi \boxtimes \omega_\tau \omega_{E/F}$ and $\pi \boxtimes \omega_\tau$ are supercuspidal representations of $GSO(V)$. Moreover, we have

$$\Theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) = \theta_\psi(\pi \boxtimes \omega_\tau \omega_{E/F}) = \tau|_{GL_2^+} \quad \text{and} \quad \theta_\psi(\pi \boxtimes \omega_\tau) = 0.$$

Proof. (i) If τ is not dihedral respect to E , we use Local-Global principle to show $\theta_\psi(\tau) \cong BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$. From [Sha90] and [GI11, Appendix A.6], there exist totally real number fields $\mathbb{F} \subset \mathbb{E}$, and a generic cuspidal representation $\pi = \otimes_v \pi_v$ on $GL_2(\mathbb{A})$ such that $\mathbb{E}_{v_0}/\mathbb{F}_{v_0} = E/F$ for some place v_0 of \mathbb{F} , we have $\pi_{v_0} = \tau$, and for other finite non-split places v of \mathbb{F} , $v \neq v_0$, π_v is a spherical representation. For the split places v , $\mathbb{E} \otimes_{\mathbb{F}} F_v \cong F_v \oplus F_v$, π_v is a spherical representation, $\omega_{\mathbb{E}_v/\mathbb{F}_v}$ is trivial. For the archimedean place, π_∞ is a discrete series representation. Fix a nontrivial character $\Psi : \mathbb{F} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ such that Ψ_{v_0} coincides with the given additive character. Let us consider the global theta lift $\theta_\Psi(\pi)$ from $GL_2^+(\mathbb{A})$ to $GSO(\mathbb{V})$, where $\mathbb{V} = \mathbb{F} \oplus \mathbb{E} \oplus \mathbb{F}$ is a 4-dimensional vector space over \mathbb{F} . Since π is generic, then the cuspidal automorphic representation $\theta_\Psi(\pi)$ is non-vanishing. We have shown $\theta_{\Psi_v}(\pi_v) \cong BC(\pi_v) \boxtimes \omega_{\pi_v} \omega_{\mathbb{E}_v/\mathbb{F}_v}$ for almost all $v \neq v_0$ in Theorem 4.9. By strong multiplicity one theorem, we have an isomorphism for the cuspidal automorphic representations $BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}} \cong \theta_\Psi(\pi)$. Hence, at local place $v = v_0$, we have

$$\theta_\psi(\tau|_{GL_2^+(F)}) = \theta(\pi)_{v_0} \cong (BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}})_{v_0} = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}.$$

(ii) Use the same strategy, set \mathbb{E}/\mathbb{F} be a totally real quadratic number field extension, and a global cuspidal automorphic representation $\pi = \otimes_v \pi_v$ on $GL_2(\mathbb{A})$ such that $\mathbb{E}_{v_0}/\mathbb{F}_{v_0} = E/F$ and $\pi_{v_0} = \tau$. Moreover, fix another special finite place v_1 , assume $\mathbb{E}_{v_1}/\mathbb{F}_{v_1}$ is a quadratic field extension and π_{v_1} is a supercuspidal representation that is not dihedral with respect to \mathbb{E}_{v_1} . Assume π' is an irreducible cuspidal automorphic representation contained in $\pi|_{GL_2^+}$, which is Ψ -generic, then the global theta lift

$$\theta_\Psi(\pi') \neq 0 \quad \text{and} \quad (\theta_\Psi(\pi'))_v \cong (BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}})_v \quad \text{for almost all } v.$$

By the strong multiplicity one theorem, one can get $\theta_\Psi(\pi') \cong BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}}$,

$$\tau^+ = \pi'_{v_0} \quad \text{and} \quad \theta_\psi(\tau^+) = \Theta_\psi(\tau^+) \cong \pi(\chi, \chi^\sigma) \boxtimes \omega_\tau \omega_{E/F}.$$

since the dimension of $Hom_N(\tau, \psi)$ is 1. Then $Hom_N(\tau^-, \psi) = 0$. Since τ^- is not ψ -generic, then $\theta_\psi(\tau^-) = 0$.

(iii) By (i) and the Kudla's result about the theta lift of a supercuspidal representation, we can get the first part. Compare with the theta lift from $GSp^+(W)$, we can obtain $\theta_\psi(\pi \boxtimes \omega_\tau) = 0$.

□

Remark 4.18. In fact, the representation τ^- participates in the theta correspondence with $GO(V_E^-)$ where $V_E^- = \epsilon V_E$ is a 2-dimensional quadratic space with quadratic form $\epsilon N_{E/F}$. By the conservation relation, the first occurrence indices of τ^- is 6 in the Witt Tower $V_r^+ = V_E \oplus \mathbb{H}^{r-1}$, i.e. the theta lift of τ^- from $GSp^+(W)$ to $GSO(V_3^+)$ is nonzero.

Remark 4.19. In the proof of (ii), we may write

$$BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}} = \theta^*(\pi) \quad \text{if} \quad BC(\pi) \boxtimes \omega_\pi \omega_{\mathbb{E}/\mathbb{F}} = \theta(\pi').$$

4.5 Local theta lift from $GSp^+(W)$ to $GSO(V^-)$

Fix $\epsilon \in F^\times$ but $\epsilon \notin NE^\times$. Now let us consider the four dimensional quadratic space V^- with a quadratic form q^- , where

$$q^-\left(\begin{pmatrix} a & x \\ \sigma(x) & d \end{pmatrix}\right) = \epsilon(N_{E/F}(x) - ad) \text{ for } v = \begin{pmatrix} a & x \\ \sigma(x) & d \end{pmatrix} \in V^-.$$

Then by Theorem 2.1, we have

$$GSO(V^-) = GSO(1, 3) \cong \frac{GL_2(E) \times F^\times}{\Delta E^\times}, \text{ where } \Delta(t) = (t, N_{E/F}(t^{-1})).$$

Similarly, we can consider the similitude dual pair $(GL_2^+, GSO(1, 3))$.

We list the results as follow without the proof.

Proposition 4.20. *Assume τ is an irreducible smooth representation of $GL_2(W)$ with infinite dimension.*

(i) *If τ is not dihedral with respect to E and $\tau \neq \pi(\omega_{E/F}, |-\cdot|^\pm) \otimes \chi$, then*

$$\Theta_{W, V^-, \psi}(\tau|_{GL_2^+}) \cong \theta_{W, V^-, \psi}(\tau|_{GL_2^+}) \cong BC(\tau) \boxtimes \omega_\tau \omega_{E/F}.$$

(ii) *If τ equals to $\pi(|-\cdot|^{1/2}, |-\cdot|^{-1/2} \omega_{E/F}) \otimes \chi$ or $\pi(|-\cdot|^{-1/2}, |-\cdot|^{1/2} \omega_{E/F}) \otimes \chi$, then*

$$\Theta_{W, V^-, \psi}(\tau|_{GL_2^+}) \cong \pi(|-\cdot|_E^{1/2}, |-\cdot|_E^{-1/2}) \otimes \chi \circ N \boxtimes \chi \circ N_{E/F} \text{ and } \theta_{W, V^-, \psi}(\tau|_{GL_2^+}) \cong \chi \circ N \boxtimes \chi \circ N.$$

(iii) *If $\tau \cong \pi(\chi, \chi \omega_{E/F})$, $\tau|_{GL_2^+} \cong \tau^+ \oplus \tau^-$, where τ^+ is ψ -generic and τ^- is ψ_ϵ -generic. Then $\theta_{W, V^-, \psi}(\tau^+) \cong 0$ and $\Theta_{W, V^-, \psi}(\tau^-) = \theta_{W, V^-, \psi}(\tau^-) \cong BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$.*

(iv) *If $\tau \cong \tau \otimes \omega_{E/F}$ is dihedral with respect to E and is a supercuspidal representation of $GL_2(F)$, then $\Theta_{W, V^-, \psi}(\tau^-) = \theta_{W, V^-, \psi}(\tau^-) \cong BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$ and $\theta_{W, V^-, \psi}(\tau^+) = 0$.*

5 Proof of Main Theorem (Local)

In this section, we give the proof of the **Main Theorem (Local)**. The key idea is to transfer the period problem $Hom_{D^\times(F)}(\pi, \mathbb{C})$, where π is an irreducible smooth representation of $D^\times(E)$, to the period problem $Hom_{PD^\times(F)}(\Sigma, \mathbb{C})$, where Σ is a representation of $GSO(V)$ associated to π . Due to Proposition 4.4, the latter one is related to the nonvanishing Whittaker model of the big theta lift $\Theta_\psi(\Sigma)$.

Recall that for $a \in F^\times$, and $y_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in V$, we have shown that

$$Hom_{SO(y_a^\perp)}(\Sigma, \mathbb{C}) \neq 0 \text{ if and only if } Hom_N(\Theta_\psi(\Sigma), \psi_a) \neq 0.$$

Proof of Main Theorem (Local) Assume $\omega_\pi|_{F^\times} = 1$, then there exists a character μ of E^\times such that $\omega_\pi = \mu^\sigma / \mu$ by Hilbert's Theorem 90.

- (1) We first prove that (i) implies (ii). Assume π is $GL_2(F)$ -distinguished. Pick $a = 1$, then y_a corresponds to the split quaternion algebra by Lemma 2.3. Due to Theorem 2.2, we have $SO(y_a^\perp, F) \cong PGL_2(F)$. Set $\Sigma = \pi \otimes \mu \boxtimes \mu|_{F^\times}$, then $\omega_\pi \cdot \mu^2 = \mu \circ N$ and

$$Hom_{PGL_2(F)}(\Sigma, \mathbb{C}) = Hom_{PGL_2(F)}(\pi \otimes \mu \boxtimes \mu|_{F^\times}, \mathbb{C}) = Hom_{GL_2(F)}(\pi \cdot \mu \circ \det, \mu|_{F^\times} \circ \det)(\dagger)$$

i.e. $Hom_{SO(y_a^\perp, F)}(\Sigma, \mathbb{C}) = Hom_{GL_2(F)}(\pi, \mathbb{C})$ is nonzero. By Proposition 4.4, we have that the representation $\Theta_\psi(\Sigma)$ of $GSp^+(W)$ is ψ -generic. By what we have shown in Section 4, we get

$$\Sigma = BC(\tau) \boxtimes \omega_{E/F} \omega_\tau,$$

i.e. $\pi \otimes \mu = BC(\tau)$ for some representation τ of $GL_2(F)$ and $\omega_\tau = \mu|_{F^\times} \omega_{E/F}$.

Conversely, if $\pi = BC(\tau) \otimes \mu^{-1}$, we can find a representation Σ of $GSO(V)$ such that the theta lift $\Theta_\psi(\Sigma)$ is ψ -generic. More precisely, if $\tau = \pi(\chi, \chi \omega_{E/F})$ is dihedral, set $\Sigma = BC(\pi(\chi, \chi)) \boxtimes \chi^2 \omega$; otherwise, set $\Sigma = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$. By Proposition 4.4, one can see that Σ is $SO(y_a^\perp)$ -distinguished, i.e. $Hom_{PGL_2(F)}(\Sigma, \mathbb{C}) \neq 0$. By the identity (\dagger) , we obtain that π is $GL_2(F)$ -distinguished.

If $\pi = BC(\tau) \otimes \mu^{-1}$, we set $\phi_\tau : WD_F \rightarrow GSp_2(\mathbb{C}) = GL_2(\mathbb{C})$ to be the Langlands parameter of τ . Assume B_τ is the non-degenerate symplectic bilinear form. Then the Langlands parameter ϕ_π with respect to π is equal to $\phi_\tau|_{WD_E} \cdot \mu^{-1}$ up to conjugacy and $\mu \mu^\sigma = \det \phi_\tau|_{WD_E}$. Assume $s \in WD_F$ and $s^2 \in WD_E^{ab}$ generates the quotient group F^\times / NE^\times . Set

$$B_\pi(m, n) = B_\tau(m, \phi_\tau(s^{-1})n).$$

It is easy to check that B_π is conjugacy-orthogonal.

Conversely, we want to show (iii) implies (ii). Since

$$(\phi_\pi \otimes \mu)^\sigma = \phi_\pi^\sigma \mu^\sigma = \phi_\pi^\vee \mu^\sigma = \phi_\mu,$$

then there is a lift $\phi_\tau : WD_F \rightarrow GL_2(\mathbb{C})$ such that $\phi_\tau|_{WD_E} = \phi_\pi \otimes \mu$, i.e. $BC(\tau) = \pi \otimes \mu$.

- (2) We use the same trick. By comparing with the theta lift from $GSO(V)$ to $GSp^+(W)$, we know $\Theta_\psi(\Sigma)$ must be ψ -generic if it is nonzero. The only trouble case is that τ is dihedral. Pick $a = \epsilon \in F^\times$ but $\epsilon \notin NE^\times$, then y_a corresponds to the non-split quaternion algebra, and $SO(y_a^\perp, F) \cong PD^\times(F)$. Then π is $D^\times(F)$ -distinguished if and only if $\Theta_\psi(\Sigma)$ is ψ_ϵ -generic for some Σ related to π . More precisely, for $\pi = BC(\tau)$, the representation only have two choice: $\Sigma = \pi \boxtimes \omega_\tau \omega_{E/F}$ or $\pi \boxtimes \omega_\tau$.

Now, we prove that (i) implies (ii). If $\Theta_\psi(\Sigma)$ is ψ_ϵ -generic, then $\Theta_\psi(\Sigma) \neq 0$ and there exists a representation τ of $GL_2(F)$ such that $\Theta_\psi(\Sigma) = \tau|_{GL_2^+}$ is irreducible. Moreover, by the proof of (1), we can get $\pi = BC(\tau) \otimes \mu^{-1}$ which is $GL_2(F)$ -distinguished. The condition that $\tau|_{GL_2^+}$ is irreducible means that τ can not be supercuspidal and dihedral with respect to E by Theorem 4.11(iii), i.e. π can not be of the form

$$\pi(\chi_1, \chi_2) \quad \text{where } \chi_1 \neq \chi_2 \text{ and } \chi_1|_{F^\times} = \chi_2|_{F^\times} = 1.$$

Conversely, if π is $GL_2(F)$ -distinguished and not in the case of Theorem 4.11(iii), then we can find a representation τ of $GL_2(F)$ such that $\tau|_{GL_2^+}$ is irreducible and participates in the theta correspondence with $GSO(V)$. Hence $\tau|_{GL_2^+}$ is ψ_ϵ -generic and it implies that π is $D^\times(F)$ -distinguished.

(3) First, (ii) and (iii) are equivalent since the group $PD^\times(F)$ is compact. Second, we prove that (iii) implies (i). Assume π is $D^\times(F)$ -distinguished and supercuspidal, then $\pi|_{GL_2(F)}$ is a projective object in the category of smooth representations of $PGL_2(F)$, then $Ext_{PGL_2(F)}^1(\pi|_{GL_2(F)}, \mathbb{C}) = 0$ and $\dim Hom_{PGL_2(F)}(\pi|_{GL_2(F)}, \mathbb{C}) = 1$. If π is the twisted Steinberg representation and is $D^\times(F)$ -distinguished, then $\pi = St_E \otimes \chi$, where $\chi|_{F^\times} = \omega_{E/F}$. Since $Ext_{PGL_2(F)}^1(\chi|_{F^\times}, \mathbb{C}) = 0$, then by the Mackey theory

$$Ext_{PGL_2(F)}^1(\pi|_{GL_2(F)}, \mathbb{C}) \cong Ext_{E^\times/F^\times}^1(\chi \circ N_{E/F}, \mathbb{C}) = 0.$$

If $\pi = \pi(\chi_1, \chi_2)$ is an irreducible principal series and is $D^\times(F)$ -distinguished, then $\chi_1\chi_2^\sigma = 1$ and by the Mackey theory, we have

$$\dim Hom_{PGL_2(F)}(\pi|_{GL_2(F)}, \mathbb{C}) = 1 + \dim Ext_{PGL_2(F)}^1(\pi|_{GL_2(F)}, \mathbb{C}).$$

Finally, if π is $GL_2(F)$ -distinguished and is not $D^\times(F)$ -distinguished, then $\pi = \pi(\chi_1, \chi_2)$ is a principal series associated with two distinct characters χ_1, χ_2 of E^\times , and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$. By the similar reasons, using the Mackey theory, we have

$$\dim Hom_{PGL_2(F)}(\pi|_{GL_2(F)}, \mathbb{C}) = \dim Ext_{PGL_2(F)}^1(\pi|_{GL_2(F)}, \mathbb{C}).$$

If a character $\mu_1 : E^\times \rightarrow \mathbb{C}^\times$ also satisfies $\omega_\pi = \mu_1^\sigma / \mu_1$ and $\pi = BC(\tau) \otimes \mu^{-1}$, we have

$$\mu_1 \mu^\sigma = \mu_1^\sigma \mu,$$

which means $\mu_1 \mu^\sigma$ is Galois invariant, hence it factors through the norm map. Assume

$$\mu_1 \mu^\sigma = \mu_F \circ N,$$

then set $\tau_1 = \tau \otimes \mu_F \mu^{-1}|_{F^\times}$, we have $\omega_{\tau_1} = \omega_{E/F} \mu_1|_{F^\times}$ and

$$\pi \otimes \mu_1 = \pi \otimes \frac{\mu_F \circ N}{\mu^\sigma} = BC(\tau) \otimes \frac{\mu_F \circ N}{\mu \mu^\sigma} = BC(\tau_1).$$

Hence, this theorem does not depend on the choice of the character μ . Then we finish the proof.

Remark 5.1. *If we consider the Euler-Poincare pairing [Pra13]*

$$EP_{PD^\times(F)}(\pi, \mathbb{C}) = \dim Hom_{PD^\times(F)}(\pi, \mathbb{C}) - \dim Ext_{PD^\times(F)}^1(\pi, \mathbb{C}),$$

then given an irreducible smooth representation π of $GL_2(E)$ with $\omega_\pi|_{F^\times} = 1$, we have

$$EP_{PGL_2(F)}(\pi|_{GL_2(F)}, \mathbb{C}) = 1 \text{ if and only if } EP_{PD^\times(F)}(\pi|_{D^\times(F)}, \mathbb{C}) = 1.$$

Remark 5.2. *For the archimedean case $F = \mathbb{R}$ and $E = \mathbb{C}$, this local main theorem for the period problem part also holds. Fix a nontrivial additive character ψ , Cognet [Cog86] proved that the theta lift of infinite dimensional representations from $GL_2^+(\mathbb{R})$ (matrix with positive determinant) to the similitude orthogonal group*

$$GSO(3, 1) \cong \frac{GL_2(\mathbb{C}) \times \mathbb{R}^\times}{\Delta \mathbb{C}^\times}$$

has the same pattern with the nonarchimedean situations. Gomez Raul and C-B Zhu [ZR] showed that for the unipotent subgroup N of $GL_2^+(\mathbb{R})$, the isomorphisms

$$\text{Hom}_N(\theta_\psi(\Sigma), \psi) \cong \text{Hom}_{PGL_2(\mathbb{R})}(\Sigma^\vee, \mathbb{C}) \text{ and } \text{Hom}_N(\theta_\psi(\Sigma), \psi_\epsilon) \cong \text{Hom}_{P\mathbb{H}^\times}(\Sigma^\vee, \mathbb{C})$$

hold for the irreducible admissible smooth representation Σ of $GSO(3, 1)$, where $\epsilon = -1$ and \mathbb{H} is the real Hamilton algebra. Then the proof to the local period for the archimedean case is almost the same.

Corollary 5.3. [Fli91] Let E be a quadratic extension of a nonarchimedean local field F . If $\pi = \pi(\chi_1, \chi_2)$ is an irreducible principal series of $GL_2(E)$, then π is $GL_2(F)$ -distinguished if and only if one of the following holds:

- $\chi_1 \chi_2^\sigma = 1$ or
- χ_1 and χ_2 are two distinct characters of E^\times with $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$.

Proof. If π is $GL_2(F)$ -distinguished, by the Main Theorem (Local), there exists a character μ and a representation τ such that $\pi = BC(\tau) \otimes \mu^{-1}$ with $\omega_\pi = \mu^\sigma / \mu$ and $\omega_\tau = \omega_{E/F} \mu|_{F^\times}$. Then $\pi(\chi_1 \mu, \chi_2 \mu)$ is Galois invariant and $\mu^\sigma = \mu \chi_1 \chi_2$. Then either $(\chi_1 \mu)^\sigma = \chi_2 \mu$ and $\chi_1 \neq \chi_2$ or $\chi_1 \mu$ and $\chi_2 \mu$ both factor through the norm map. For the first case, we have

$$\chi_1|_{F^\times} = \chi_2|_{F^\times} \text{ and } \mu|_{F^\times} = \omega_\tau \cdot \omega_{E/F} = (\chi_1 \mu)|_{F^\times} \omega_{E/F} \cdot \omega_{E/F}, \text{ i.e. } \chi_1|_{F^\times} = 1.$$

For the second case, we have $(\chi_1 \mu)^\sigma = \chi_1 \mu$, then $\chi_1^\sigma \chi_2 = 1$.

Conversely, if χ_1 and χ_2 are trivial on F^\times , set $\pi = \pi(\frac{\mu_1^\sigma}{\mu_1}, \frac{\mu_2^\sigma}{\mu_2})$ and $\mu = \mu_1 \mu_2$. Since $\chi_1 \neq \chi_2$, then the character $\mu_1 \mu_2^\sigma$ of E^\times does not factor through the norm. Let τ to be the dihedral supercuspidal representation of $GL_2(F)$ with respect to $(E, \mu_1 \mu_2^\sigma)$. Then $\omega_\tau = \mu|_{F^\times} \omega_{E/F}$ and $\pi = BC(\tau) \otimes \mu^{-1}$ is $GL_2(F)$ -distinguished. If $\chi_1 \chi_2^\sigma = 1$, set $\mu = \chi_2^\sigma$ and $\tau = \pi(\omega_{E/F}, \chi_2|_{F^\times})$, then $\pi = BC(\tau) \otimes \mu^{-1}$ is $GL_2(F)$ -distinguished. \square

Corollary 5.4. For the twisted Steinberg representation $\pi = St_E \otimes \chi$ of $GL_2(E)$, the following statements are equivalent:

- (i) π is $GL_2(F)$ -distinguished;
- (ii) π is $D^\times(F)$ -distinguished;
- (iii) $\chi|_{F^\times} = \omega_{E/F}$.

More general, we can consider the conditions for $(GL_2(F), \chi)$ -period problems, i.e.

$$\text{Hom}_{GL_2(F)}(\pi, \chi).$$

Proposition 5.5. Let χ be a character of F^\times . Assume π is a representation of $GL_2(E)$ and $\omega_\pi|_{F^\times} = \chi^2$. Then the following statements are equivalent:

- (i) π has a nonzero $(GL_2(F), \chi)$ -period;
- (ii) $\pi = BC(\tau) \otimes (\chi_E \cdot \mu^{-1})$ for a triple (τ, χ_E, μ) , where μ and χ_E are characters of E^\times and τ is a representation of $Gl_2(F)$ satisfying

- $\chi_E|_{F^\times} = \chi$;

- $\omega_\pi = \chi_E^2 \mu^\sigma / \mu$ and
- $\omega_\tau = \omega_{E/F} \cdot \mu|_{F^\times}$.

Proof. Set $\pi' = \pi \otimes \chi_E^{-1}$ where $\chi_E|_{F^\times} = \chi$. Then $\omega_{\pi'}|_{F^\times} = 1$ and

$$\text{Hom}_{GL_2(F)}(\pi, \chi) = \text{Hom}_{GL_2(F)}(\pi', \mathbb{C})$$

which has been discuss in the Main Theorem (Local). \square

6 Global Theta Lift for Similitude Groups

Let F be a number field. Assume E is a quadratic field extension of F . Let \mathbb{A} be the adele ring of F and $\mathbb{A}_E = \mathbb{A} \otimes E$. Fix a unitary additive character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$. Let W be a symplectic vector space over F and let $V = E \oplus \mathbb{H}$ be a 4-dimensional quadratic space over F . Let ω_ψ be the Weil representation for the dual pair $Sp(W, \mathbb{A}) \times SO(V, \mathbb{A})$.

Recall

$$R = GSp^+(W) \times GSO(V) \text{ and } R_0 = \{(g, h) \in R \mid \lambda_W(g) \cdot \lambda_V(h) = 1\}$$

and the action are the same as in the local setting, see Section 3.

For a Schwartz function $\phi \in S(V, \mathbb{A})$ and $(g, h) \in R_0(\mathbb{A})$, set

$$\theta_\psi(\phi)(g, h) = \sum_{x \in V(F)} \omega_\psi(g, h) \phi(x).$$

Then $\theta_\psi(\phi)$ is a function of moderate growth on $R_0(F) \backslash R_0(\mathbb{A})$. Suppose $\pi \boxtimes \mu$ is a cuspidal automorphic representation of $GSO(V, \mathbb{A})$ and $f \in \pi \boxtimes \mu$, we set

$$\theta_\psi(\phi, f)(g) = \int_{SO(V, F) \backslash SO(V, \mathbb{A})} \theta_\psi(\phi)(g, h_1 h) \cdot \overline{f(h_1 h)} dh$$

where h_1 is any element of $GSO(V, \mathbb{A})$ such that $\text{sim}_V(h_1) = \text{sim}_W(g^{-1})$. Then

$$\Theta_\psi(\pi \boxtimes \mu) = \left\{ \theta_\psi(\phi, f) : \phi \in S(V, \mathbb{A}), f \in \pi \boxtimes \mu \right\}$$

is an automorphic representation (possibly zero) of $GSp(W)^+(\mathbb{A})$. Consider the Fourier coefficient of $\theta_\psi(\phi, f)$ with respect to ψ_a , where $\psi_a(x) = \psi(ax)$, and a is an arbitrary nonzero element in \mathbb{A} , we have

$$\begin{aligned} & Wh_{N, \psi_a}(\theta_\psi(\phi, f)) \\ &= \int_{N(F) \backslash N(\mathbb{A})} \overline{\psi_a(u)} \int_{SO(V, F) \backslash SO(V, \mathbb{A})} \theta_\psi(\phi)(u, h) \cdot \overline{f(h)} dh du \\ &= \int_{SO(V, F) \backslash SO(V, \mathbb{A})} \overline{f(h)} \cdot \int_{N(F) \backslash N(\mathbb{A})} \overline{\psi(au)} \cdot \sum_{x \in V(F)} \omega_\psi(u, h) \phi(x) du dh \\ &= \int_{SO(V, F) \backslash SO(V, \mathbb{A})} \overline{f(h)} \cdot \sum_{x \in V^a} \phi(h^{-1}x) dh \end{aligned}$$

where $V^a = \{x \in V(F) \mid q(x) = a\} = SO(V, F) \cdot \{y_a\}$ and the stabilizer of y_a is $SO(y_a^\perp, F)$. Hence, we have

$$\begin{aligned} Wh_{N, \psi_a}(\theta_\psi(\phi, f)) &= \int_{SO(V, F) \backslash SO(V, \mathbb{A})} \overline{f(h)} \cdot \sum_{\gamma \in SO(y_a^\perp, F) \backslash SO(V, F)} \phi(h^{-1} \gamma^{-1} y_a) dh \\ &= \int_{SO(y_a^\perp, F) \backslash SO(V, \mathbb{A})} \overline{f(h)} \phi(h^{-1} y_a) dh \\ &= \int_{SO(y_a^\perp, \mathbb{A}) \backslash SO(V, \mathbb{A})} \phi(h^{-1} y_a) \cdot \int_{SO(y_a^\perp, F) \backslash SO(y_a^\perp, \mathbb{A})} \overline{f(th)} dt dh \end{aligned}$$

That means $\theta_\psi(\phi, f)$ has non-zero (N, ψ_a) -period if and only if

$$\int_{SO(y_a^\perp, F) \backslash SO(V, \mathbb{A})} \overline{f(h)} \phi(h^{-1} y_a) dh \neq 0.$$

We define the period integral

$$P_{SO(y_a^\perp)}(f) := \int_{SO(y_a^\perp, F) \backslash SO(y_a^\perp, \mathbb{A})} f(t) dt,$$

from [Gan, Proposition 5.2], we know that $Wh_{N, \psi_a}(\theta_\psi(\phi, f))$ is nonzero for some f and ϕ if and only if the period integrate $P_{SO(y_a^\perp)}$ is nonzero on π . We repeat the proof as follow.

Proposition 6.1. *The ψ_a -coefficient of $\Theta_\psi(\pi)$ is nonzero if and only if the period integral $P_{SO(y_a^\perp)}$ is nonzero on π . In particular, if $P_{SO(y_a^\perp)}$ is nonzero on π for some a , then the global theta lift $\Theta_\psi(\pi)$ is nonzero.*

Proof. If $P_{SO(y_a^\perp)} = 0$ on π , then $Wh_{N, \psi_a} = 0$. If $P_{SO(y_a^\perp)}$ is nonzero on π , then we may choose $f \in \pi$ such that the function $h \mapsto P_{SO(y_a^\perp)}(h \cdot f)$ is a nonzero function on $SO(V_a^\perp, \mathbb{A})$. Since

$$Wh_{N, \psi_a}(\theta_\psi(\phi, f)) = \int_{SO(y_a^\perp, \mathbb{A}) \backslash SO(V, \mathbb{A})} \phi(h^{-1} y_a) \cdot \overline{P_{SO(y_a^\perp)}(h \cdot f)} dh,$$

we need to show that one can find ϕ such that the above integral is nonzero. But note that $V^a \subset V$ is a Zariski-closed subset, and

$$SO(y_a^\perp) \backslash SO(V) \cong V^a \quad \text{as varieties defined over } \mathbb{F}.$$

So it suffices to show that the restriction map $S(V, \mathbb{A}) \rightarrow S(V^a, \mathbb{A})$ is a surjection.

For each local field F_v , the surjectivity of $S(V, F_v) \rightarrow S(V^a, F_v)$ is clear. But the adele statement has an additional subtlety. Namely, the spaces $S(V, \mathbb{A})$ and $S(V^a, \mathbb{A})$ are restricted tensor products $\otimes'_v S(V, F_v)$ and $\otimes'_v S(V^a, F_v)$ with respect to a family of distinguished vectors $\{\phi_v^\circ\}$ and φ_v° for almost all v . We need to take note that the restriction map takes ϕ_v° to φ_v° for almost all v . For the case at hand, we can choose a base \mathcal{B} of V which endows V and V^a with an integral structure. For almost all v , we may take ϕ_v° and φ_v° to be the characteristic functions of $V(\mathcal{O}_v)$ and $V^a(\mathcal{O}_v)$ respectively. The result then follows from the fact that

$$V^a(\mathcal{O}_v) = V^a(F_v) \cap V(\mathcal{O}_v)$$

then we are done. \square

Corollary 6.2. *If $\theta_\psi(\phi, f) = 0$, then $P_{SO(y_a^\perp)}$ vanishes on π .*

Automorphic realization of Mixed Model Recall, we have a Witt decomposition

$$W = X + Y \text{ and } V = Fv_0 + V_E + Fv_0^*$$

where $V_E = E$ is the quadratic space with a quadratic form $e \mapsto N_{E/F}(e)$. The stabilizer $Q = TU$ of v_0 is a Borel subgroup of $SO(V)$, with

$$T = GL(Fv_0) \times E^1 \text{ and } U \subset Hom(v^*, V_E).$$

We can consider the maximal isotropic subspace

$$\mathbb{X} = W \otimes v_0^* + Y \otimes V_E \subset W \otimes V$$

and write an element of \mathbb{X} as

$$(w, e) = (x, y, e) \in W + V_E = X + Y + V_E.$$

Assume ω_ψ is the Weil representation of $Sp(W, \mathbb{A}) \times SO(V, \mathbb{A})$, then extend it to $R_0(\mathbb{A})$. And the action of $GSp^+(W, \mathbb{A}) \times GSO(V, \mathbb{A})$ is defined as before, See Section 4.1.

Given a cuspidal representation τ of $GSp^+(W)$, set $f \in \tau$. For $\phi' \in S(X \otimes V, \mathbb{A})$, assume the partial Fourier transform function is $\phi = I(\phi') \in S(\mathbb{X}, \mathbb{A})$. Define

$$\theta_\psi(\phi, f) = \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \overline{f(g_1 g)} \sum_{x \in \mathbb{X}(F)} \omega_\psi(g_1 g, h) \phi(x) dg,$$

where $\lambda_W(g_1) = \lambda_V(h)$. Then $\theta_\psi(\phi, f)$ is an automorphic representation of $GSO(V)(\mathbb{A})$, the unipotent radical $U(\mathbb{A}) \cong \mathbb{A}_E$. Consider the subset $\{(x, 0, x^{-1})\} \subset (W - \{0\}) + V_E$, then

$$\begin{aligned} & Wh_{U, \psi_E}(\theta_\psi(\phi, f)) \\ &= \int_{U(F) \backslash U(\mathbb{A})} \overline{\psi_E(u)} \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \theta_\psi(\phi)(u, g) \cdot \overline{f(g)} dg du \\ &= \int_{U(F) \backslash U(\mathbb{A})} \overline{\psi_E(u)} \cdot \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \overline{f(g)} \cdot \sum_{x \in W + V_E} \omega_\psi(u, g) \phi(x) dg du \\ &= \int_{U(F) \backslash U(\mathbb{A})} \overline{\psi_E(u)} \cdot \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \overline{f(g)} \cdot \\ &\quad \sum_{(x, y, v) \in W + V_E} \psi(-xyN(u) + y \cdot \text{tr}(u\sigma(v))) \cdot (\omega_0(g)\phi(g^{-1} \begin{pmatrix} x \\ y \end{pmatrix}))(v - uy) dg du \\ &= \int_{U(F) \backslash U(\mathbb{A})} \overline{\psi_E(u)} \cdot \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \overline{f(g)} \cdot \sum_{\gamma \in N(F) \backslash Sp(W, F)} \\ &\quad \sum_{v \in V_E} \psi(-xyN(u)) \psi_E(y \cdot u\sigma(v)) (\omega_0(g)\phi(g^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \gamma))(v) dg du \\ &= \int_{U(F) \backslash U(\mathbb{A})} \overline{\psi_E(u)} \cdot \int_{N(F) \backslash Sp(W, \mathbb{A})} \overline{f(g)} \cdot \sum_{v \in V_E} \psi_E(u\sigma(v)) (\omega_0(g)\phi(g^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}))(v) dg du \\ &= \int_{N(F) \backslash Sp(W, \mathbb{A})} \overline{f(g)} (\omega_0(g)\phi(g^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}))(1) dg \\ &= \int_{N(\mathbb{A}) \backslash Sp(W, \mathbb{A})} \omega_\psi(g) \phi(1, 0, 1) \cdot \int_{N(F) \backslash N(\mathbb{A})} \overline{f(n(x)g)} \psi(x) dx dg \end{aligned}$$

Corollary 6.3. *If τ is ψ -generic, then $\theta_\psi(\phi, f)$ is ψ_E -generic.*

The proof is very similar with the proof in Proposition 6.1.

Assume $\Sigma = \pi \boxtimes \chi$ is an irreducible cuspidal representation of $GSO(V, \mathbb{A})$, where π is a cuspidal automorphic representation of $GL_2(\mathbb{A}_E)$. Let S be a finite set, containing archimedean places of F , and the places v where E_v is ramified or Σ_v is ramified.

Proposition 6.4. *The following statements are equivalent:*

- (i) *the theta lift $\theta_\psi(\Sigma)$ from $GSO(V, \mathbb{A})$ of $GSp^+(W, \mathbb{A})$ is nonzero;*
- (ii) *$\Sigma = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$ for some automorphic cuspidal representation τ of $GL_2(\mathbb{A})$;*
- (iii) *the partial L -function $L^S(1, \Sigma, \text{Std}) = \infty$ and $\theta_{\psi_v}(\Sigma_v) \neq 0$ for all $v \in S$.*

Moreover, in which case, the cuspidal representation $\tau^+ = \theta_\psi(\Sigma)$ of $GL_2^+(\mathbb{A})$ is ψ -generic and then Σ is $PGL_2(\mathbb{A})$ -distinguished.

Proof. First, we prove that (iii) implies (i). Assume $L^S(s, \Sigma, Std)$ has a pole at $s = 1$. Given $f_1, f_2 \in \Sigma$ and $\phi_1, \phi_2 \in S(V, \mathbb{A})$, we have the double zeta integral

$$\begin{aligned} Z(s, f_1, f_2, \Phi) &= \int_{SO(V, F)^2 \backslash SO(V, \mathbb{A})^2} \overline{f_1(h)} f_2(h_2) E((h_1, h_2), s, \Phi) dh_1 dh_2 \\ &= \prod_v Z_v(s, f_1, f_2, \Phi_v) \cdot L^S(s + \frac{1}{2}, \Sigma, Std) := Z_S(s, f_1, f_2, \Phi_S) L^S(s + \frac{1}{2}, \Sigma, Std) \end{aligned}$$

where $E((h_1, h_2), s, \Phi)$ is the Eisenstein series of $SO_8(\mathbb{A})$ associated to the standard section Φ , and the local zeta integral

$$Z_v(s, f_1, f_2, \Phi_v) = \int_{SO(V, F_v)} \Phi_v(h, 1) \overline{\langle \Sigma_v(h) f_{1,v}, f_{2,v} \rangle_v} dh.$$

From [GT11a], by the regularized Siegel-Weil formula, we can get

$$\langle \theta_\psi(f_1, \phi_1), \theta_\psi(f_2, \phi_2) \rangle = c \operatorname{Res}_{s=\frac{1}{2}} L^S(s + \frac{1}{2}, \Sigma, Std) \cdot Z_S(\frac{1}{2}, f_1, f_2, \Phi)$$

where c is a nonzero constant. Since $\theta_{\psi_v}(\Sigma_v) \neq 0$ for all $v \in S$, then there exist $f_1^\circ, f_2^\circ \in \Sigma$, and $\phi_1, \phi_2 \in S(V)$ such that $Z_S(\frac{1}{2}, f_1^\circ, f_2^\circ, \Phi_S) \neq 0$, and the theta lifting $\theta_\psi(\Sigma)$ is non-vanishing.

Now, we show (i) implies (ii). If Σ participates in the theta correspondence with $GSp^+(W, \mathbb{A})$, then $\theta_{\psi_v}(\Sigma_v) \neq 0$ and $\Sigma_v \cong BC(\tau_v) \boxtimes \omega_{\tau_v} \omega_{E_v/F_v}$ at each place v of F . Assume $\tau \subset \Theta_\psi(\Sigma)$ is the cusp constituent of $GL_2^+(\mathbb{A})$, which is globally ψ -generic. By the multiplicity one theorem, we have $\Sigma \cong BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$. Conversely, if $\Sigma = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$, we want to show $\theta_\psi(\pi) \neq 0$, where π is an irreducible component of $\Sigma|_{SO(4)}$. Then we are done, which is due to the following double see-saw diagram

$$\begin{array}{ccc} Sp(W) \times Sp(W) & & SO(\mathbb{V}) \\ & \searrow \quad \swarrow & \\ \Delta Sp(W) & & SO(V^-) \times SO(V^+) \end{array}$$

then $\langle \theta_\psi(\phi_1, f_1), \theta_\psi(\phi_2, f_2) \rangle$

$$= \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \int_{SO(V, F)^2 \backslash SO(V, \mathbb{A})^2} \theta_\psi(\phi_1)(g, h_1) \overline{f_1(h_1)} \cdot \overline{\theta_\psi(\phi_2)(g, h_2)} f_2(h_2) dh_1 dh_2 dg,$$

due to the regularized Siegel-Weil formula identity, and it is equal to

$$\begin{aligned} &\int_{SO(V, F)^2 \backslash SO(V, \mathbb{A})^2} \overline{f_1(h_1)} f_2(h_2) \int_{Sp(W, F) \backslash Sp(W, \mathbb{A})} \theta_{\psi, \mathbb{V}, W}(\phi_1 \otimes \bar{\phi}_2)(g, (h_1, h_2)) dg dh_1 dh_2 \\ &= c \cdot \operatorname{Res}_{s=\frac{1}{2}} Z(s, f_1, f_2, \Phi) = c Z_S(\frac{1}{2}, f_1, f_2, \Phi_S) \cdot \operatorname{Res}_{s=\frac{1}{2}} L^S(s + \frac{1}{2}, \Sigma, Std). \end{aligned}$$

By the local results, one has $\theta_{\psi_v}(\Sigma_v) \neq 0$, which implies that $Z_v(\frac{1}{2}, f_{1,v}, f_{2,v}, \Phi_v) \neq 0$ for some $f_{1,v}, f_{2,v}$ and Φ_v when $v \in S$. Since

$$L^S(s + \frac{1}{2}, \Sigma, Std) = \zeta_F^S(s + \frac{1}{2}) L^S(s + \frac{1}{2}, \tau, Ad \otimes \omega_{E/F}),$$

let $s = 1/2$, then $L(1, \tau, Std \otimes \omega_{E/F})$ is nonvanishing by Shahidi's result [Sha81]. Hence $L^S(1, \Sigma, Std) = \infty$, and then $\theta_\psi(BC(\tau) \boxtimes \omega_\tau \omega_{E/F})$ is non-vanishing.

From (ii) to (iii), if $\Sigma = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$, then for each v , Σ_v participates in the theta correspondence with $GL_2^+(F_v)$, and $\theta_{\psi_v}(\Sigma_v)$ is ψ_v -generic. Hence $\theta_\psi(\Sigma)$ is globally ψ -generic. If $\Sigma = BC(\tau) \boxtimes \omega_\tau \omega_{E/F}$, then

$$L^S(s, BC(\tau) \boxtimes \omega_\tau \omega_{E/F}, Std) = L^S(s, \sigma, Ad \otimes \omega_{E/F}) \zeta_F^S(s).$$

By the result of Shahidi [Sha81], $L^S(s, \sigma, Ad \otimes \omega_{E/F})$ is nonzero at $s = 1$. Then $L^S(s, \Sigma, Std)$ has a pole at $s = 1$.

From the proof above, we know $\tau^+ = \theta_\psi(\Sigma) = \theta_\psi(BC(\tau) \boxtimes \omega_\tau \omega_{E/F})$ is ψ -generic. By Proposition 6.1, we pick $a = 1$ and $y_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in V$, then $SO(y_a^\perp, \mathbb{A}) \cong PGL_2(\mathbb{A})$ and Σ is $PGL_2(\mathbb{A})$ -distinguished. \square

Remark 6.5. *The local theta lift from $SL(2)$ to $O(4)$ is the same as the local theta lift from $SL(2)$ to $SO(4)$. And there is a key result in the proof: the local zeta integral $Z_v(\frac{1}{2})$ does not vanish on $Im\Phi_v \otimes \bar{\pi} \otimes \pi$ if and only if $\Theta_{\psi_v, W_v, V_v}(\pi_v) \neq 0$. This will play an important role in the proof of our Main Theorem (Global).*

7 Proof of the Main Theorem (Global)

In this section, we give the proof of the **Main Theorem (Global)**.

Let us recall the notations. Let E be a quadratic extension of a number field F . Let W be a 2-dimensional symplectic space over F . Let D be a quaternion algebra over F . Let $Z(\mathbb{A}) = \mathbb{A}^\times$ be the center of $D^\times(\mathbb{A})$. Let π^D be a cuspidal automorphic irreducible representation of $D^\times(\mathbb{A}_E)$, with central character ω_{π^D} and $\omega_{\pi^D}|_{\mathbb{A}^\times} = 1$. Assume the Jacquet-Langlands correspondence representation π of π^D is a cuspidal automorphic representation of $GL_2(\mathbb{A}_E)$. Let S to be a finite set of places containing archimedean places in F , such that for all $v \notin S$, π_v^D is unramified and E_v is unramified. Since $D \otimes_F E$ may not be $M_2(E)$, we can not find a point in $V = V_E \oplus \mathbb{H}$ corresponding to D by Lemma 2.3. Then we need another 4-dimensional quadratic space over F .

Set $X_D = \{x \in D \otimes_F E | x^* = \sigma(x)\}$, consider the theta lifting $\theta_\psi(\phi, f)$ from $GSO(X_D, \mathbb{A})$ to $GSp^+(W, \mathbb{A})$. Pick $y = 1 \otimes 1 \in X_D$, then $SO(y^\perp, F) \cong PD^\times(F)$ by Theorem 2.2. Let $\Sigma^D = \pi^D \otimes \mu \boxtimes \mu|_{\mathbb{A}^\times}$ be the cusp form on $GSO(X_D, \mathbb{A})$. Respectively, we set $\Sigma = \pi \otimes \mu \boxtimes \mu|_{F^\times}$ to be the cusp form of $GSO(V, \mathbb{A})$. For $(g, t) \in D^\times(\mathbb{A}_E) \times \mathbb{A}^\times$ and $f \in \Sigma^D$, we define

$$f(g, t) = f^D(g) \mu(N(g)) \cdot \mu(t), \text{ where } f^D \in \pi^D.$$

Then we have $f(d, N(d)^{-1}) = f^D(d)$ and

$$\int_{Z(\mathbb{A}) D^\times(F) \backslash D^\times(\mathbb{A})} f^D(t) dt = \int_{PD^\times(F) \backslash PD^\times(\mathbb{A})} f(h) dh.$$

From Proposition 6.1, one can see that Σ^D is $PD^\times(\mathbb{A})$ -distinguished if and only if $\theta_\psi(\Sigma^D)$ is ψ -generic. In fact, the only difference between Σ^D and Σ is at the local places v where D_v ramified and E_v split. By Proposition 4.3, Σ_v is $PD_v^\times(F_v)$ -distinguished if and only if $\Sigma_v = \tau \boxtimes \tau^\vee$ for some representation τ of $D^\times(F_v)$. So the key point is the local results at the places v where D_v ramified and E_v non-split, $X_{D,v} \cong V^-$, which have been classified in Section 5.

Proof of the Main Theorem (Global). Since $\omega_{\pi^D}|_{Z(\mathbb{A})} = 1$, by Hilbert's Theorem 90, the central character $\omega_{\pi^D} = \mu^c/\mu$ for some grossencharacter μ of \mathbb{A}_E^\times .

It is clear that Σ^D is $PD^\times(\mathbb{A})$ -distinguished if and only if π^D is $D^\times(\mathbb{A})$ -distinguished.

Now we prove (ii) implies (iii). Assume Σ^D is $PD^\times(\mathbb{A})$ -distinguished, then $\theta_{X_D, W, \psi}(\Sigma^D)$ has a nonzero (N, ψ) -Whittaker model due to Proposition 6.1, i.e. $\tau = \theta_{X_D, W, \psi}(\Sigma^D) \neq 0$, which implies $L^S(1, \Sigma^D, Std) = \infty$. Because of the Rallis inner product and the regularized Siegel-Weil formula, we have

$$0 \neq \langle \theta_\psi(f_1, \phi_1), \theta_\psi(f_2, \phi_2) \rangle = c \cdot \text{Res}_{s=\frac{1}{2}} L^S(s + \frac{1}{2}, \Sigma^D) Z^S(s, f_1, f_2, \Phi_S), \text{ where } c \neq 0$$

for some $f_1, f_2 \in \Sigma^D$, and $\phi_1, \phi_2 \in S(V, \mathbb{A})$. Moreover $\theta_{W, V, \psi}(\tau)$ is a cuspidal representation of $GSO(V, \mathbb{A})$ and coincides with Σ at almost all local places. By the strong multiplicity one theorem of $GSO(V, \mathbb{A})$, we get $\Sigma = \theta_{W, V, \psi}(\tau)$. By the local Howe duality, $\tau_v = \theta_{V, W, \psi}(\Sigma_v)$ is ψ_v -generic and then Σ_v is $PGL_2(F_v)$ -distinguished. By the local results, for each place v of F where D_v is ramified and E_v is not split, we have $\Sigma_v^D = \Sigma_v$ and

$$0 \neq \text{Hom}_N(\theta_{X_D, v, W, \psi_v}(\Sigma_v^D), \psi_v) = \text{Hom}_N(\theta_{V_v, W, \psi_v}(\Sigma_v), \psi_{v, \epsilon_v}), \text{ where } \epsilon_v \in F_v - NE_v^\times.$$

This means Σ_v can be written as $BC(\tau_0) \otimes \omega_{\tau_0} \omega_{E/F}$ for some representation τ_0 of $GL_2(F_v)$ and $\tau_0|_{GL_2^+}$ is irreducible.

Finally, we prove that (iii) implies (ii). Assume the local conditions hold. By the local results, we know $\theta_{X_D, v, W, \psi_v}(\Sigma_v^D) \neq 0$ and is both ψ_{v, ϵ_v} -generic and ψ_v -generic for each v where D_v ramified, E_v non-split and $\epsilon_v \in F_v^\times - NE_v^\times$. If Σ is $PGL_2(\mathbb{A})$ -distinguished, then $\tau = \theta_{V, W, \psi}(\Sigma)$ is ψ -generic as a cuspidal representation of $GSp^+(W)$ and the partial L-function $L^S(s, \Sigma, Std)$ has a pole at $s = 1$. By the Rallis inner product, we can obtain $\Sigma^D = \theta_{W, X_D, \psi}(\tau)$ is a nonzero cuspidal representation of $GSO(X_D, \mathbb{A})$ because the only possible troublesome cases have been removed by the local assumptions. Then Σ^D is $PD^\times(\mathbb{A})$ -distinguished by Proposition 6.1 and τ is ψ -generic. \square

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